MATH 13 FALL 2019 HOMEWORK 7

Due: Thursday, December 5, 2019 Please turn in at the discussion.

If you work in a group of two please turn in only one paper, but put both names and Student ID's on the paper. If you work alone, turn in a paper with your name and Student ID only.

Student name/id (include all students in the group):

IMPORTANT INSTRUCTIONS:

- It is crucial that you write your arguments and explanations clearly and that each argument clearly shows how you arrive at the conclusions from the assumptions. This is the point of homeworks – to practice understanding of the material, proofwriting, and the ability to express your understanding.
- Try to write your arguments as efficiently as possible, meaning that you should judge what to write and what not. Writing irrelevant text makes the argument confused and difficult to understand. Leaving out important points makes the argument incomplete. So you need to judge what is relevant and what not. There is no recipe for this; the ability to recognize what to write and what not is a skill which needs to be developed, and this course (= Math 13) is intended to help you with this.
- When preparing the homeworks, please follow the **Rules for homeworks** on the course website under **Course information and policies**.

ALSO IMPORTANT: In arguments and proofs you are writing, **never** use symbols $\therefore$ and $\because$ but instead, write everything as a text.

Also avoid using the symbol $\nexists$.

1. (5pt) Consider a binary relation R on A. Recall that E_A is the equality relation on A, that is, $E_A = \{\langle a, a\rangle \mid a \in A\}$. Prove the following:

(a) R is reflexive iff $E_A \subseteq R$.
(b) R is symmetric iff $R^{-1} = R$.
(c) R is transitive iff $R \circ R \subseteq R$.
(d) If $R \subseteq R^{-1}$ then $R^{-1} = R$.

Remark. Recall that we view binary relations as sets. So to show that binary relations R_1, R_2 are equal you need to prove that

$$\langle x, y\rangle \in R_1 \iff \langle x, y\rangle \in R_2$$

for every ordered pair $\langle x, y\rangle$.

2. (5pt) For each of the following functions decide whether it is injective, surjective, bijective. Justify your conclusions by proofs.

(a) $f_1 : (0, 1) \to (1, \infty)$ is defined by $f_1(x) = \frac{1}{x}$.
(b) Consider open intervals (a, b) and (c, d). Function $f_2 : (a, b) \to (c, d)$ is defined by

$$f_2(x) = c + \frac{d-c}{b-a}(x - a)$$

(c) $f_3 : \mathbb{R} \to \mathbb{R}^+$ is devined by $f(x) = \frac{1}{1+x^2}$.
(d) $f_4 : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

(e) Given are bijections $g : A \to C$ and $h : B \to D$. Function $f_5 : A \times B \to C \times D$ is defined by

$$f_5(\langle x, y\rangle) = \langle g(x), h(y)\rangle$$

Remark. Recall that we consider functions as sets. So if f is a function then $y = f(x)$ means $\langle x, y\rangle \in f$.

3. (5pt) Recall that if E is an equivalence relation on A then for each $x \in A$ the set $[x] = \{y \in A \mid \langle y, x\rangle \in E\}$ is the equivalence class of x with respect to E. Also recall that the quotient A/E is the set of all equivalence classes $[x]$ where x runs over the entire A. Notice that we can define a function $k : A \to A/E$ called the quotient map as follows:

$$k(x) = [x].$$

(a) Prove that k is surjective.
(b) Prove that k is injective iff $E = E_A$ (See Problem 1 for the definition of E_A).

Now consider a function $f : A \to B$. Define a binary relation E_f on A by

$$\langle x, y\rangle \in E_f \iff f(x) = f(y)$$

(c) Prove that E_f is an equivalence relation on A.

Analogously as above, we have the quotient map $k_f : A \to A/E_f$ defined by $k_f(x) = [x]_{E_f}$. In what follows we will write briefly $[x]$ instead of $[x]_{E_f}$. Define a binary relation $i \subseteq A/E_f \times B$ as follows:

$$\langle [x], y\rangle \in i \iff y = f(x)$$

(d) Prove that i is a function and $\mathrm{dom}(i) = A/E_f$. Also prove that i is injective.

Remark. First formulate precisely what is to be proved. Notice that you have proved that $i : A/E_f \to B$, and that i is a well-defined function, in the sense explained in the lecture: The value $i([x])$ does not depend of the representative of the equivalence class $[x]$.

(e) Prove that $i \circ k_f = f$.

Remark. You can base your proof on the Remark in Problem 1. Recall however, that if $g : X \to Y$ and $h : Y \to Z$ then $(h \circ g)(x) = h(g(x))$ for all $x \in X$, which will give you a simpler proof.

4. (5pt) Consider a set A. Define a binary relation E on $\mathcal{P}(A)$ as follows:

$$\langle X, Y \rangle \in E \iff \text{There exists a bijection } f : X \to Y$$

Prove that E is an equivalence relation on $\mathcal{P}(A)$. Also, if A is finite, how many equivalence classes with respect to E there are? Explain how you arrived at your answer.