MATH 13 FALL 2019 MIDTERM PRACTICE PROBLEMS

Below are some sample problems similar to those which will be given on the midterm.

1. Here we consider sets $A, B, C \subseteq U$ where $U$ is a universal set.
   (a) If $A \subseteq B$ determine the relationship between $A^c$ and $B^c$.
   (b) If $A \Delta B \subseteq C$ determine the relationship between $A \setminus C$ and $A \cap B$.
   (c) Determine $P(A) \cap P(A^c)$.
   (d) Is $P(A) \cup P(A^c) = P(U)$?
   (e) Is $P(A) \setminus P(B) \subseteq P(A \setminus B)$?

2. Express the following statements in the symbolic language using symbols as instructed in Problem group 1 and then negate the statement. Use only symbols $=, \leq$ and arithmetic operations like $+, -, \cdot$ and similar. In (e) consider a set $A \subseteq \mathbb{Q}$.
   (a) Every positive integer can be expressed as a sum of three squares of integers.
   (b) For every real number $r$ the equation $8x + r = 0$ has precisely one real solution.
   (c) There are at least three different integers in the interval $(-2, 2)$.
   (d) For every real number $x$ larger than 1 the value $1/x$ is smaller than 1 but larger than 0.
   (e) Between any two distinct integers there is an element of $A$.

3. Prove the following statements. In each case choose a suitable type of proof.
   (a) If $a, b$ are odd integers then $a^2 - b^2$ is divisible by 4.
   (b) If $p$ is an odd integer then the equation $x^3 + x + p = 0$ does not have an integer solution.
   (c) If $a, b$ are integers divisible by an integer $n$ then the product $a \cdot b$ is divisible by $n^2$.
   (d) If $a, b, c, d$ are four consecutive integers then the sum $a + b + c + d$ is even.
   (e) The product of three consecutive integers is always divisible by 6.

4. Prove the following by induction on $n$.
   (a) For each $n > 0$, $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4} n^2 (n + 1)^2$.
   (b) For each $n > 0$, $17^n - 13^n$ is divisible by 4.
   (c) For every $n > 0$, $1 + a + a^2 + \cdots + a^n = (a^{n+1} - 1)/(a - 1)$.
   (d) For every $n > 0$, $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + a^1b^{n-2} + b^{n-1})$.
      It would be nicer to use the "$\sum$"-notation:
      \[ a^n - b^n = (a - b) \cdot \sum_{k=0}^{n-1} a^{n-1-k} b^k \]
   (e) For every $n > 1$, $(2n + 1)!$ is divisible by $2^n \cdot n!$.