FINAL-SAMPLE

1. (4pt) Perform the following computations.

(a) **(2pt)** Use Euclid's algorithm to compute the greatest common divisor of 246 and 602.

(b) **(2pt)** Express the number 502 in the canonical form as given by the Fundamental Theorem of Arithmetic.

Hint. Do hand-computations. In (b) there is no standard method. Just do an ad hoc computation.

2. (4pt) The following two questions concern the greatest common divisor.

(a) **(2pt)** You have a balance scale and weights that are 6oz and 8oz. Is it possible to check the weight of an object of weight 22oz? If so, explain how. If not, prove that it is not possible.

(b) **(2pt)** Let $a = b \cdot q + s$. Prove that $\text{g.c.d}(a, b) = \text{g.c.d}(b, s)$.

Hint. In (a) recall that $\text{g.c.d}(a, b)$ is a linear combination of a, b. In (b) do the straightforward verification of definition.

3. (5pt) Let R be a binary relation.

(a) **(1pt)** Prove: $A \subseteq B \implies R[A] \subseteq R[B]$.

(b) **(1pt)** Assume $A \subsetneqq B$. Does it follow that $R[A] \subsetneqq R[B]$? If so, prove it. If not, give an example of R, A, B that demonstrates this is false.

(c) **(1pt)** Give an example of R, A, B that demonstrates that

$$R[A \cap B] \neq R[A] \cap R[B].$$

(d) **(2pt)** Prove:

$$R[\bigcup_{i \in I} A_i] = \bigcup_{i \in I} R[A_i]$$

Hint. These are straightforward applications of the respective definitions.

4. (4pt) Let R be the equivalence relation on $\mathbb{N}$. defined as follows:

$$\langle m, n\rangle \in R \iff \text{the number } n \text{ has } m \text{ digits.}$$

(a) **(2pt)** Is R^{-1} a function? Prove or disprove.

(b) **(2pt)** Let S be the inequality on $\mathbb{N}$, i.e.

$$\langle p, q\rangle \in S \iff p < q.$$

Prove that $S \circ R \subseteq S$. Is $S \circ R = S$? Prove or disprove.

5. (8pt) The following questions concern injectivity/surjectivity of functions. In (a) – (d) decide whether the function in question is injective/surjective. Justify your answers: If the function is injective/surjective, prove this. If the function fails to be injective/surjective, disprove this.

(a) **(2pt)** $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $f(\langle m, n\rangle) = m^n$.

(b) **(2pt)** $g : (-1, 1) - \{0\} \to \mathbb{R} - \{0\}$ is defined by $g(x) = 1/x$.

(c) **(2pt)** $h : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ is defined by $h(X) = \min(X)$.

(d) **(2pt)** $k : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A)$ is defined by $k(X, Y) = X - Y$. Here $A \neq \varnothing$.

6. (6pt) The last two problems concern equivalence relations.

(a) **(3pt)** R is a binary relation on $\mathbb{R} \times \mathbb{R}$ defined as follows:

$$\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \in R \iff (x - x' \in \mathbb{Z} \;\&\; y - y' \in \mathbb{Z}).$$

Prove that R is an equivalence relation on $\mathbb{R} \times \mathbb{R}$. Describe the equivalence classes in geometrical terms.

(b) **(3pt)** A is the set of all two-element subsets of $\mathbb{Z}$. S is a binary relation on A defined by

$$\langle a, b\rangle \in S \iff a, b \text{ contain the same number of even numbers}$$

Prove that S is an equivalence relation on A. How many equivalence classes are there? Name them.