GRADED HOMEWORK 1

1. (4pt) When we divide an integer n by 5, the possible remainders are $0, 1, 2, 3, 4$. Obviously, if the remainder is 0 then n is divisible by 5. In the remaining cases, i.e. when the remainder is $1, 2, 3$ or 4, the number n is not divisible by 5.

To state this rigorously, we say that any integer n is of the form $5k$ or $5k + 1$ or $5k + 2$ or $5k + 3$ or $5k + 4$ where k is an integer.

Prove the following fact: n is divisible by 5 if and only if $n^4 - 1$ is not divisible by 5.

Hint. Consider the five possibilities for n as described above.

2. (1pt) In the lecture we proved: The product $x \cdot y$ of two integers is even if and only if at least one of them is even.

Use this fact to prove the following: The product $x \cdot y \cdot z$ of three integers is even if and only if at least one of them is even.

Hint. When we compute the product of three integers $x \cdot y \cdot z$, we first compute the product of **two** integers $x \cdot y$ and then again compute the product of **two** integers, this time the product of $x \cdot y$ and z.

3. (2pt) In the lecture we proved that $\sqrt{2}$ is irrational. **Immitate the same proof** to show that $\sqrt[3]{2}$ is irrational.

Hint. In the proof that $\sqrt{2}$ is irrational we used Corollary 2.6 from the lecture. This time you use the result from Problem 2 above.

4. (3pt) Let $a, b > 0$ be rational numbers. Show that $a\sqrt{2}+b\sqrt{3}$ is irrational.

Hint. Try a proof by contradiction. Assume that $a\sqrt{2} + b\sqrt{3}$ is rational; then try to express $\sqrt{2}$ using rational expressions that involve only rational numbers. Finally apply the fact from the lecture that $\sqrt{2}$ is irrational to get a contradiction.