GRADED HOMEWORK 3

1. **(2pt)** Prove by induction that for all $n > 0$ we have

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2.$$

2. **(1+1pt)** Define a sequence of numbers $a_1, a_2, a_3, \ldots$ by recursion as follows:

- $a_1 = a_2 = 1$
- $a_n = 2a_{n-1} + 3a_{n-2}$.

Use mathematical induction to prove:

(a) If $n > 2$ then $a_n > 3^{n-2}$.

(b) If $n > 2$ then $a_n < 2 \cdot 3^{n-2}$.

3. **(3pt)** Prove by induction on n that for all $n > 0$, the arithmetic mean of 2^n positive real numbers is at least as large as the geometric one. That is, prove that if $a_1, a_2, \ldots, a_{2^n}$ are positive real numbers then

$$\frac{1}{2^n}(a_1 + a_2 + \cdots + a_{2^n}) \geq \sqrt[2^n]{a_1 \cdot a_2 \cdot \cdots \cdot a_{2^n}}.$$

Hint. Do the case $n = 1$ carefully. For larger $n+1$, express the arithmetic mean of 2^{n+1} many numbers as the arithmetic mean of two arithmetic means of 2^n many numbers. Then apply the induction hypothesis for n and also for $n = 1$.

Remark. The above inequality holds for all natural numbers, not only for those of the form 2^n, but the proof is a bit more involved.

4. **(3pt)** Prove by induction on n the so-called Dirichlet (or pigeonhole) principle: If $n > 1$ and we split a finite set with at least n elements into strictly less than n disjoint nonempty pieces then at least one of those pieces has more than one element.

Hint. First observe that "at least n" can be replaced by n, and it is practical to consider the set $\{1, 2, 3, \ldots, n\}$. Then proceed by induction on n. At step $n \mapsto n+1$, split the set $\{1, 2, 3, \ldots, n, n+1\}$ into less than $n+1$ pieces and discuss the two cases: either $\{n+1\}$ is one of the pieces or otherwise.