GRADED HOMEWORK 4

1. (2pt) Use induction to prove that every nonempty finite set of natural numbers has a **largest** element.

Hint. Do induction on the number of elements of sets. When doing the induction step $n \mapsto n+1$, write down carefully the statement you intend to prove. Here is the strategy: Given an $(n+1)$-element set of natural numbers, remove one element. You obtain an n-element set of natural numbers. Explain in detail how you get the largest element of the original $(n+1)$-element set once you know the largest element of the n-element set that you obtained by removing one element.

2. (2+2pt) Given is a natural number $b > 0$. Prove that every $z \in \mathbb{N}$ can be **uniquely** expressed in the form

$$a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots + a_n \cdot b^n.$$

where $0 \leq a_k < b$ for all $k \in \{0, \ldots, n\}$ and $a_n \neq 0$. That is, prove that given $z \in \mathbb{N}$, numbers $n, a_0, \ldots, a_n$ as above exist and are **unique** in the following sense: If

$$a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots + a_n \cdot b^n = a'_0 + a'_1 \cdot b + a'_2 \cdot b^2 + \cdots + a'_m \cdot b^m$$

where $0 \leq a_k < b$ for all $k \in \{0, \ldots, n\}$, $0 \leq a'_k < b$ for all $k \in \{0, \ldots, m\}$, $a_n \neq 0$ and $a_m \neq 0$ then $m = n$ and $a_k = a'_k$ for all $k \in \{0, \ldots, n\}$.

The above way of expressing integers is called b-adic expansion, and b is called the **base** of expansion. In practice, the most commonly used expansions are decadic (here $b = 10$) and dyadic, or binary (here $b = 2$).

Hint. Proceed by induction, but use the strong version of induction: Assuming that the result on expansions holds for all $z' < z$ prove that the result holds for z. Here you have to use the division algorithm (Theorem 4.1 in the lecture).

3. (4pt) If a, b, c are integers then the greatest common divisor of a, b, c is the unique $d \in \mathbb{N}$ satisfying the following two conditions:

(a) d is a commod divisor of a, b, c, i.e. $d \mid a$, $d \mid b$ and $d \mid c$.

(b) If d' is any common divisor of a, b, c then $d' \mid d$.

Notice that (a) implies that $d > 0$. Prove:

(i) **(1pt)** d is really unique, that is: If d_1 and d_2 are natural numbers that satisfy (a) and (b) then $d_1 = d_2$.

(ii) **(2 + 1pt)** If at least one of the numbers a, b, c is distinct from 0 then the largest common divisor of a, b, c exists.

Hint. (i) follows from the definition of the greatest common divisor by a straightforward argument. concerning (ii), imitate the proof of Theorem 4.5 from the lecture. This is the theorem on greatest common divisor of two numbers.