GRADED HOMEWORK 5
Due: Thursday, November 29

1. (6pt) If $a, b \neq 0$ are two integers then the **smallest common multiple** is a number $m \in \mathbb{N}^+$ satisfying the following two requirements:

(i) m is a common multiple of a, b, that is $a \mid m$ and $b \mid m$.

(ii) If $m' \neq 0$ is a common multiple of a, b then $m \mid m'$.

Prove the following facts concerning the smallest common multiple of a, b where $a, b \neq 0$.

(a) **(1pt)** The smallest common multiple of a, b is unique. That is, if m_1 and m_2 are smallest common multiples of a, b then $m_1 = m_2$.

(b) **(2pt)** The smallest common multiple of a, b exists and in fact it is the least element of the set

$$M = \{z \in \mathbb{N}^+ \mid \ a \mid z \ \& \ b \mid z\}.$$

In other words, the smallest common multiple is also least among all positive multiples of a, b with respect to the natural ordering of $\mathbb{N}$.

(c) **(1pt)** Let d be the greatest common divisor of a, b. Then $\frac{a}{d}$ and $\frac{b}{d}$ are relative primes.

(d) **(2pt)** Let d be the greatest common divisor of a, b and m be the smallest common multiple of a, b. Prove that $m = \frac{a \cdot b}{d}$.

Hint. (a) should be proved directly form (i) and (ii) by a straightforward argument. This is similar to Problem 3(i) in Homework 4.

Concerning (b), let m be the least element of M. First show that m exists. Then you have to verify that m satisfies (i) and (ii). (i) should be clear to you. Regarding (ii): Use the division algorithm. Express m' in the form $m' = q \cdot m + r$ where $0 \leq r < m$ and argue that $r = 0$. For this, show that r is a common multiple of a, b. If you don't know how to start, look at the proof of the theorem on greatest common divisors; this is Theorem 4.5 in the lecture (in the book the theorem is labelled as Theorem 3.10).

(c) Use Theorem 4.5 from the lecture to express the greatest common divisor of d as a linear combination of a and b.

(d) Show that $\frac{a \cdot b}{d}$ satisfies (i) and (ii). (i) should be clear to you. Concerning (ii), use Theorem 4.5 in the lecture. Find $x, y \in \mathbb{Z}$ so that $d = a \cdot x + b \cdot y$. Then multiply this equation by m'. On the right side try to recover the product $a \cdot b$; here you use the fact m' is a common multiple of a, b.

2. (4 × 1pt) Prove/establish the following facts about binary relations.

(a) Let R be a binary relation and A be a set. Prove:

$$R[A] = \bigcup_{a \in A} R[\{a\}].$$

(b) Let R be a binary relation and A be a set. Prove that $A \subseteq R[R^{-1}[A]]$.

(c) Let

$$\begin{aligned} R &= \{\langle m, n\rangle \mid m, n \in \mathbb{N}^+ \;\&\; m \mid n\} \\ S &= \{\langle m, n\rangle \mid m, n \in \mathbb{N}^+ \;\&\; m \text{ is a multiple of } n\}. \end{aligned}$$

Determine R^{-1} and S^{-1}.

(d) Let A be a set and let

$$\begin{aligned} R &= \{\langle x, X\rangle \mid x \in A \;\&\; X \subseteq A\} \\ S &= \{\langle X, Y\rangle \mid X, Y \subseteq A \;\&\; X \subseteq Y\}. \end{aligned}$$

Determine $S \circ R$.

Hint. These are straightforward verifications of the definitions.