GRADED HOMEWORK 6
Due: Friday, December 7

1. (4 × 1pt) Prove/establish the following facts about functions.

(a) Let $f : A \to B$ be a function and $X_i \subseteq A$ for all $i \in I$.

$$f[\bigcap_{i\in I} X_i] \subseteq \bigcap_{i\in I} f[X_i].$$

(b) Let $f : A \to B$ and $g : B \to C$. If both f, g are injective then $g \circ f$ is injective.

(c) Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective then f is injective.

(d) Let $f : A \to B$. If f is injective then there is a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.

Hint. (a),(b) and (c) are direct verifications of the corresponding definitions. For (a) you should also recall the definition of union. In (d), you have to construct the function g. The way you define g is actually forced by the requirement $g \circ f = \mathrm{id}_A$; your task is to find out what this requirement says.

2. (2 × 1pt) In either of the follwing two cases decide whether the function F and G in question is injective and surjective. Justify your answers.

(a) $\mathcal{A}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$. The function $F : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined as follows:

$$F(\langle f, g\rangle) = \begin{cases} \text{the unique function } h : \mathbb{R} \to \mathbb{R} \text{ satisfying} \\ h(x) = f(x) + g(x) \text{ for all } x \in \mathbb{R}. \end{cases}$$

(b) $\mathcal{A} = \{X \in \mathcal{P}(\mathbb{N}) \mid 0 \notin X\}$. The function $G : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathbb{N})$ is defined as follows:

$$G(X) = X \cup \{0\}.$$

Remark. These are straightforward verifications of definitions.

3. (2 × 2pt) In either case determine whether the binary relation in question is an equivalence relation or an ordering relation. If the relation is an equivalence relation, describe its equivalence classes, if possible in geometric terms.

(a) A is the set of all two-element subsets of $\mathbb{N}$. R is a binary relation on A defined as follows:

$$\langle \{a, b\}, \{a', b'\} \rangle \in R \iff a + b \leq a' + b'.$$

(b) S is the binary relation on $\mathbb{R} \times \mathbb{R}$ (i.e. on the Euclidean plane) defined as follows:

$$\langle \langle x, y \rangle, \langle x', y' \rangle \rangle \in S \implies \max\{|x|, |y|\} = \max\{|x'|, |y'|\}.$$