MATH 141 – SAMPLE FINAL

1. (5pt) Let \( f : \ell^1 \rightarrow \mathbb{R} \) be defined by
\[
f(a) = \sum_{k=0}^{\infty} a_k
\]
where \( a = (a_0, a_1, \ldots, a_k, \ldots) \). Show that the map \( f \) is continuous. (Recall that the metric on \( l^1 \) is given by
\[
d(a, b) = \sum_{k=0}^{\infty} |a_k - b_k|.
\]
The metric on \( \mathbb{R} \) is the standard one.)

2. (5pt) Consider the following sequence \( \langle a_n \mid n \in \mathbb{N} \rangle \) in the Baire space \( \mathcal{N} \). Each \( a_n \) is of the form
\[
a_n = (a_{n,0}, a_{n,1}, a_{n,2}, \ldots a_{n,k} \ldots)
\]
where
\[
a_{n,k} = k \text{ if } k \leq n \text{ and } a_{n,k} = n \text{ otherwise .}
\]
Determine whether the sequence \( \langle a_n \mid n \in \mathbb{N} \rangle \) is convergent in the Baire space \( \mathcal{N} \), and in the affirmative case find its limit.

3. (5pt) Consider the following sequence \( \langle b_n \mid n \in \mathbb{N} \rangle \) in the Baire space \( \mathcal{N} \). Each \( b_n \) is of the form
\[
b_n = (b_{n,0}, b_{n,1}, b_{n,2}, \ldots b_{n,k} \ldots)
\]
where \( b_{n,k} \) is even for at least one \( k \). Assume that
\[
b = (b_0, b_1, b_2, \ldots b_k \ldots)
\]
is the limit of \( \langle b_n \mid n \in \mathbb{N} \rangle \). Can we infer that at least one \( b_k \) is even?

4. (5pt) Let \( (X, d) \) be the metric space \( C^\infty([0, 1]) \) and \( (X', d') \) be the metric space \( C^\infty([0, 2]) \). Thus, elements of \( X \) are continuous functions \( f : [0, 1] \rightarrow \mathbb{R} \).
and elements of $X'$ are continuous functions $g : [0, 2] \to \mathbb{R}$. In both cases, the metric is the sup metric. Show that the map $h : (X, d) \to (X', d')$ defined by

$$h(f)(x) = f \left( \frac{x}{2} \right)$$

is a homeomorphism between $(X, d)$ and $(X', d')$.

5. (5pt) Let $(X, d)$ be the space of all infinite sequences of natural numbers

$$a = (a_0, a_1, \ldots, a_k, \ldots)$$

such that $a_k \leq k$ for all $k \in \mathbb{N}$. The metric is the same as in the Baire space, that is if $a \neq b$ are two distinct sequences in $X$ then

$$d(a, b) = \frac{1}{n + 1}$$

where $n$ is the least with the property that $a_n \neq b_n$. Prove that $(X, d)$ is a complete metric space.

6. (5pt) Let $D$ be the set of all infinite sequences of real numbers $a = (a_0, a_1, \ldots, a_k, \ldots)$ that are eventually 0, that is there is $n_0$ such that $a_k = 0$ for all $k \geq n_0$. Prove that $D$ is a dense subset of $\ell^1$. This includes proving that each sequence from $D$ is an element of $\ell^1$. 
