1. (a) \((4pt)\) \(S^*\) denotes the set of all finite nonempty sequences of natural numbers. Let \(\langle p_i \mid i \in \mathbb{N} \rangle\) be the increasing enumeration of all prime numbers. The map \(f : S^* \rightarrow \mathbb{N}\) given by
\[
f(\langle a_0, a_1, \ldots, a_n \rangle) = p_0^{a_0+1} \cdot p_1^{a_1+1} \cdots p_n^{a_n+1}
\]
is injective. Proof: Assume \(f(\langle a_0, a_1, \ldots, a_n \rangle) = f(\langle b_0, b_1, \ldots, b_m \rangle)\), that is
\[
p_0^{a_0+1} \cdot p_1^{a_1+1} \cdots p_n^{a_n+1} = p_0^{b_0+1} \cdot p_1^{b_1+1} \cdots p_m^{b_m+1}.
\]
We want to show that \(\langle a_0, a_1, \ldots, a_n \rangle = \langle b_0, b_1, \ldots, b_m \rangle\).

We first show that \(m = n\). If \(n < m\) then the left side in (1) is divisible by \(p_m\), since \(a_m + 1 > 0\). But the right side is not divisible by \(p_m\). This is impossible, since they are equal. For similar reasons we cannot have \(m < n\). It follows that \(m = n\).

Next we show that \(a_i = b_i\) whenever \(0 \leq i \leq n\). If there is some \(i\) such that \(a_i < b_i\), divide both sides of (1) by \(p_i^{a_i+1}\). We get
\[
p_0^{a_0+1} \cdot p_1^{a_1+1} \cdots p_{i-1}^{a_{i-1}+1} \cdot p_i^{a_{i+1}+1} \cdots p_n^{a_n+1} = p_0^{b_0+1} \cdot p_1^{b_1+1} \cdots p_i^{b_i-a_i} \cdots p_n^{b_n+1}.
\]
So if \(a_i < b_i\) then \(b_i - a_i > 0\), and therefore the right side in (2) is divisible by \(p_i\), whereas the left side is not. Again, this is impossible since they are equal. Similarly we conclude that \(b_i < a_i\) is also impossible. It follows that \(a_i = b_i\) for all \(i\).

Now since \(n = m\) and \(a_i = b_i\) for all \(i \in \{0, \ldots, n\}\), we conclude that the two sequences are equal. So we have proved:
\[
f(\langle a_0, a_1, \ldots, a_n \rangle) = f(\langle b_0, b_1, \ldots, b_m \rangle) \implies \langle a_0, a_1, \ldots, a_n \rangle = \langle b_0, b_1, \ldots, b_m \rangle,
\]
which shows that \(f\) is injective. But then \(f[S^*]\) is an infinite set of natural numbers, and therefore \(f[S^*]\) is countable (by the facts from the discussions).

Since \(f\) is injective, it is a bijection between \(S^*\) and \(f[S^*]\). Thus, since \(f[S^*]\) is countable, \(S^*\) must be countable as well. So we have a bijection \(g' : \mathbb{N} \rightarrow S^*\). Now \(S\) is obtained just by adding \(\emptyset\) to \(S^*\), and therefore must be countable as well. For instance, we can define a bijection \(g : \mathbb{N} \rightarrow S\) as follows:
\[
g(n) = \begin{cases} 
\emptyset & \text{if } n = 0 \\
g'(n - 1) & \text{if } n > 0
\end{cases}
\]
(b) (2pt) We know from (a) that the set $S$ of all finite sequences of natural numbers is countable. To see that $S_A$ is countable, we construct a bijection between $S$ and $S_A$. Since $A$ is countable, there is a bijection $h : \mathbb{N} \to A$. We use this bijection to define a map $H : S \to S_A$ as follows:

$$H(\emptyset) = \emptyset$$

$$H(\langle a_0, a_1, \ldots, a_n \rangle) = \langle h(a_0), h(a_1), \ldots, h(a_n) \rangle.$$ 

We show that $H$ is a bijection between $S$ and $S_A$. For this we have to verify that $H$ is both injective and surjective.

Proof of injectivity: Assume $H(\langle a_0, a_1, \ldots, a_n \rangle) = H(\langle b_0, b_1, \ldots, b_m \rangle)$. This means that $\langle h(a_0), h(a_1), \ldots, h(a_n) \rangle = \langle h(b_0), h(b_1), \ldots, h(b_m) \rangle$. This implies that $m = n$ and $h(a_i) = h(b_i)$ for all $i \in \{0, \ldots, n\}$. But since $h$ is injective, $h(a_i) = h(b_i)$ implies $a_i = b_i$ for each $i$. Thus we have $m = n$ and $a_i = b_i$ for all $i \in \{0, \ldots, n\}$. We can conclude that $\langle a_0, a_1, \ldots, a_n \rangle = \langle b_0, b_1, \ldots, b_m \rangle$. This proves the injectivity of $H$.

Proof of surjectivity: Let $\langle b_0, b_1, \ldots, b_m \rangle$ be a sequence from $S_A$. We want to find some sequence $\langle a_0, a_1, \ldots, a_n \rangle$ from $S$ such that

$$H(\langle a_0, a_1, \ldots, a_n \rangle) = \langle b_0, b_1, \ldots, b_m \rangle.$$ 

First, $H(\emptyset) = \emptyset$ by the definition of $H$, so $\emptyset$ is in the range of the map $H$. From now on we assume that $\langle b_0, b_1, \ldots, b_m \rangle$ is not the empty sequence. Since $h : \mathbb{N} \to A$ is a bijection, to each $i \in \{0, \ldots, m\}$ there is some $a_i$ such that $h(a_i) = b_i$. Then $H(\langle a_0, a_1, \ldots, a_n \rangle) = \langle h(a_0), h(a_1), \ldots, h(a_m) \rangle = \langle b_0, b_1, \ldots, b_m \rangle$. Here the first equality follows from the definition of $H$ and the second equality follows from our choice of the numbers $a_i$. This proves that $H$ is surjective.

2. (a) (2pt) First, the pointwise limit. If $x \neq 0$ then $x^2 > 0$ and so $1 + x^2 > 1$. Thus if $n \to \infty$ then also $(1 + x^2)^n \to \infty$ and so $\frac{1}{(1 + x^2)^n} \to 0$. If $x = 0$ we have $\frac{1}{(1 + x^2)^n} = 1$ for all $n$ and so $\frac{1}{(1 + x^2)^n} \to 1$. Thus, the pointwise limit $f$ is defined by

$$f(x) = \begin{cases} 
0 & \text{if } x \neq 0 \\
1 & \text{if } x = 0.
\end{cases}$$

The function sequence $f_n$ does not converge to $f$ uniformly on $\mathbb{R}$: If we let e.g. $\varepsilon = \frac{1}{2}$ then we can find arbitrarily large $n \in \mathbb{N}$ with the property $|f_n(x_n) - f(x_n)| \geq \frac{1}{2}$ for some $x_n \in \mathbb{R}$. In fact, we get this for all $n \in \mathbb{R}$.
(although this much is not needed for the failure of uniform convergence.)

By solving the inequality
\[
\frac{1}{(1 + x^2)^n} > \frac{1}{2},
\]
(you should solve it as a part of your answer to the question I will not do this here, as this is straightforward) we see that it is enough to let

\[x_n = \sqrt{2\pi} - 1.\]

Then \( |f_n(x_n) - f(x_n)| = \frac{1}{\sqrt{(1 + \sqrt{2\pi} - 1)^n}} - 0 = \frac{1}{2}. \)

(b) (2pt) We show that \( f \) is not uniformly continuous on the interval \((0, 1)\). It suffices to show that for each \( \delta > 0 \) we can find \( x, y \in (0, 1) \) such that \( |x - y| \leq \delta \) and \( |g(x) - g(y)| \geq 1 \). Since \( \sin(2\pi n) = 0 \) and \( \sin(2\pi n + \pi/2) = 1 \) for all \( n \in \mathbb{N} \), it is enough to find \( n \) large enough that

\[
\frac{1}{2\pi n} - \frac{1}{2\pi n + \pi/2} \leq \delta \tag{3}
\]

and let \( x = \frac{1}{2\pi n} \) and \( y = \frac{1}{2\pi n + \pi/2} \). Then

\[
|g(x) - g(y)| = |\sin(1/x) - \sin(1/y)| = |\sin(2\pi n) - \sin(2\pi n + \pi/2)| = |0 - 1| = 1.
\]

To get \( n \) as in (3), simplify the left side of (3) (again, I will not do it here, but you should do it in your homework) and get

\[
\frac{1}{2\pi n(4n + 1)} \leq \delta,
\]

and so \( 2\pi n(4n + 1) \geq \frac{1}{\delta} \). But since \( 2\pi n(4n + 1) > n \), it is enough to choose \( n \) such that \( n \geq \frac{1}{\delta} \).