WEEK 1

1. (a) (4pt) Show that the set $S$ of all finite sequences of natural numbers is countable.

   Hint. We consider $\emptyset$ a finite sequence, the so called empty sequence. $S^*$ denotes the set of all finite nonempty sequences of natural numbers. So the only difference between $S^*$ and $S$ is that $S$ has the empty sequence as an element, whereas $S^*$ not. Let $\langle p_i \mid i \in \mathbb{N} \rangle$ be the increasing enumeration of all prime numbers. Consider the map $f : S^* \to \mathbb{N}$ given by

   $$f((a_0, a_1, \ldots, a_n)) = p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot \ldots \cdot p_n^{a_n+1}.$$

   (b) (2pt) Consider any countable set $A$. Let $S_A$ denote the set of all finite sequences of elements of $A$. Show that $S_A$ is countable.

   Hint. Try to find a bijection between $S_A$ and the set $S$ from (a).

2. (a) (2pt) Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

   $$f_n(x) = \frac{1}{(1+x^2)^n}.$$

   Determine the pointwise limit $f$ of this sequence. Does the sequence $\langle f_n \rangle_n$ converge uniformly to $f$ on $\mathbb{R}$? Give rigorous arguments supporting your conclusions.

   (b) (2pt) Consider the function $g : (0, 1) \to (0, 1)$ defined by

   $$g(x) = \sin \frac{1}{x}.$$

   Is this function uniformly continuous on $(0, 1)$? Give a rigorous argument supporting your conclusion.