1. **(2+2pt)** (a) We have to verify that \(d'\) satisfies the requirements in the definition of a metric.

First, \(d'(x, y) = d(f(x), f(y)) \geq 0\) since \(d\) is a metric. Also, \(d'(x, y) = 0\) iff \(d(f(x), f(y)) = 0\) iff \(f(x) = f(y)\) iff \(x = y\). The second equivalence follows from the fact that \(d\) is a metric, the last one from the fact that \(f\) is injective.

Second, \(d'\) is symmetric: \(d'(x, y) = d(f(x), f(y)) = d(f(y), f(x)) = d'(y, x)\). The middle equality follows again from the symmetricity of \(d\), since \(d\) is a metric.

Third, the triangle inequality. Let \(x, y, z \in Y\). Then
\[
d'(x, z) = d'(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) = d'(x, y) + d'(y, z).
\]
Again, the inequality here is the triangle inequality for \(d\).

(b) If \(f\) is not injective then \(d'\) is not a metric on \(Y\), since there are \(x \neq y\) such that \(f(x) = f(y)\). Then \(d'(x, y) = d(f(x), f(y)) = 0\), which is not possible if \(d'\) were a metric. Here \(d(f(x), f(y)) = 0\) since \(f(x) = f(y)\) and \(d\) is a metric on \(X\).

2. **(2pt)** First, \(d\) is really a function with values in \(\mathbb{R}\), since
\[
\sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} \leq \sum_{n=0}^{\infty} \frac{1}{2^{2n}} = 2,
\]
which tells us that the series
\[
\sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n}
\]
with nonnegative members is convergent.

Obviously,
\[
d(a, b) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} \geq \sum_{n=0}^{\infty} \frac{0}{2^n} = 0.
\]
Also, \(d(a, b) = 0\) iff
\[
\sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} = 0
\]
iff \(|a_n - b_n| = 0\) for all \(n \in \mathbb{N}\), which in turn holds iff \(a_n = b_n\) for all \(n \in \mathbb{N}\). The last statement is equivalent to \(a = b\). This verifies the first axiom of a metric.

Second, we verify the symmetricity:

\[
d(a, b) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} = \sum_{n=0}^{\infty} \frac{|b_n - a_n|}{2^n} = d(b, a).
\]

Finally we verify the triangle inequality. Let \(a, b, c \in X\). Then

\[
d(a, c) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} = \sum_{n=0}^{\infty} \frac{|(a_n - b_n) + (b_n - c_n)|}{2^n} \leq \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n} + \sum_{n=0}^{\infty} \frac{|b_n - c_n|}{2^n} = d(a, b) + d(b, c).
\]

The second inequality follows from the fact that \(|u + w| \leq |u - v| + |v - w|\) for all \(u, v, w\).

3. (a) (1pt) \(\mathbb{Q} \cap (\sqrt{2}, \sqrt{3})\) is open in \(\mathbb{Q}\). Proof: If \(x \in (\sqrt{2}, \sqrt{3})\), let \(\delta\) be the smaller of \(x - \sqrt{2}\) and \(\sqrt{3} - x\). If \(z \in B(x, \delta)\) then \(x - \delta < z < x + \delta\). But \(x - \delta \geq \sqrt{2}\) since \(\delta \leq x - \sqrt{2}\), and \(x + \delta \leq \sqrt{3}\) since \(x + \delta \leq \sqrt{3}\). Hence \(\sqrt{2} < z < \sqrt{3}\). This proves that \(B(x, \delta) \subseteq (\sqrt{2}, \sqrt{3})\), and thereby shows that \(\mathbb{Q} \cap (\sqrt{2}, \sqrt{3})\) is open in \(\mathbb{Q}\).

Similarly as before we show that both intervals \((-\infty, \sqrt{2})\) and \((\sqrt{3}, \infty)\) are open in \(\mathbb{Q}\). So their union, call it \(G\), is open in \(\mathbb{Q}\), too. But \(G\) is precisely the complement of \((\sqrt{2}, \sqrt{3})\) in \(\mathbb{Q}\), so \((\sqrt{2}, \sqrt{3})\) is closed in \(\mathbb{Q}\).

(b) (2pt) Let \(G \subseteq (0, 1)\) be open. We show that if \(x \in G\) then

\[
x \in \left(\frac{k - 1}{2^n}, \frac{k + 1}{2^n}\right) \subseteq G
\]

for some suitable \(k < 2^n\). This will show that \(G\) can be expressed as the union of intervals of the required form.

Since \(G\) is open, there is some \(\varepsilon > 0\) such that \((x - \varepsilon, x + \varepsilon) \subseteq G\). Pick \(n\) large enough that \(\frac{1}{2^n} < \varepsilon/2\). Let \(k\) be smallest such that \(k < 2^n\) and

\[
k \geq x. \tag{1}
\]
Such a $k$ exists, since letting $k = 2^n$ we get $\frac{k}{2^n} = 1 > x$. Then

$$x - \varepsilon < x - \varepsilon/2 < x - \frac{1}{2^n} \leq \frac{k - 1}{2^n} < x; \quad (2)$$

the second inequality here follows from our choice of $n$, the second one from (1), and the third one from the fact that $k$ is the smallest satisfying (1).

Since $\frac{k-1}{2^n} < x$ as we proved in (2), we have

$$\frac{k + 1}{2^n} = \frac{k - 1}{2^n} + 2 \cdot \frac{1}{2^n} < x + 2 \cdot (\varepsilon/2) = x + \varepsilon. \quad (3)$$

Putting the results from (2) and (3) together, we get

$$x - \varepsilon < \frac{k - 1}{2^n} < x \leq \frac{k}{2^n} < \frac{k + 1}{2^n} < x + \varepsilon.$$

This tells us that $\left(\frac{k-1}{2^n}, \frac{k+1}{2^n}\right) \subseteq (x - \varepsilon, x + \varepsilon)$, what we intended to prove.