WEEK 8

1. (5pt) Let \((X, d)\) be a complete metric space and \(F \subseteq X\). Prove that the following are equivalent:

(a) The subspace \((F, d)\) is complete.

(b) The set \(F\) is closed in \((X, d)\).

(Points: (a)⇒(b) is worth 2pt; (b)⇒(a) is worth 3pt).

Hint. (a)⇒(b): Show that if \((x_n)_n\) is a sequence of elements in \(F\) and \(x_n \to x\) in the sense of \((X, d)\) then \(x \in F\). Check that \((x_n)_n\) is Cauchy in the sense of \((F, d)\) and utilize this fact together with the completeness of \((F, d)\).

(b)⇒(a): You need to show that if \((x_n)_n\) is a Cauchy sequence in \((F, d)\) then \((x_n)_n\) has a limit in \((F, d)\). Use the completeness of \((X, d)\) to show that \((x_n)_n\) has a limit in \((X, d)\) and then argue that this limit must be in \(F\).

2. (5pt) Let \((X_1, d_1), \ldots, (X_n, d_n)\) be metric spaces. Recall that the Cartesian product \((X, d)\) of these spaces is defined as follows:

\[
X = X_1 \times X_2 \times \ldots \times X_n
\]

\[
d(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}
\]

where \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) are elements of \(X\). Prove:

(a) (2pt) If all spaces \((X_1, d_1), \ldots, (X_n, d_n)\) are separable then also \((X, d)\) is separable.

(b) (3pt) If all spaces \((X_1, d_1), \ldots, (X_n, d_n)\) are complete then also \((X, d)\) is complete.

Hint. (a) For each \(i\) pick a countable dense set \(D_i\) for \((X_i, d_i)\) and using these dense sets construct a countable dense set \(D\) for \((X, d)\).

(b) Consider any Cauchy sequence \((x_m)_m\) in \((X, d)\). Each \(x_m\) is of the form \((x_{m,1}, \ldots, x_{m,n})\) where \(x_{m,i} \in X_i\). Show that if you fix \(i \in \{1, \ldots, n\}\) then each sequence \((x_{m,i})_m\) is Cauchy in \((X_i, d_i)\). Then appeal to the completeness of \((X_i, d_i)\) and argue that each sequence \((x_{m,i})_m\) has a limit in \((X_i, d_i)\); denote this limit by \(x_i\). Finally show that \(x = (x_1, \ldots, x_n)\) is a limit of \((x_m)_m\) in the space \((X, d)\).