1. (5pt) Let $F$ be the set of all rational functions $f$ of the form

$$f(x) = \frac{ax + b}{cx + d}$$

with rational coefficients $a, b, c, d$. Show that $F$ is countable.

You can use the following two facts: (1) the set of all rational numbers $\mathbb{Q}$ is countable, and (2) if we have finitely many countable sets $A_1, \ldots, A_n$ then the Cartesian product $A_1 \times \ldots \times A_n$ is countable.

2. (5pt) Given is the set $X$ of all linear functions $f_a : \mathbb{R} \to \mathbb{R}$ of the form $f_a(x) = ax$ where $a \in \mathbb{R}$. Show that the function $d$ defined by

$$d(f_a, f_b) = |a - b|$$

is a metric on $X$.

3. Consider the space $BF(\mathbb{R})$ of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ with the sup metric. Let $Z \subseteq BF(\mathbb{R})$ be the set of all functions $f \in BF(\mathbb{R})$ with the property $f(0) = f(1)$.

(a) (2pt) Is $Z$ open?

(b) (3pt) Is $Z$ closed?

Apply the definition of openness and closedness directly (but of course you can use a different approach, if you prefer).

4. (5pt) Work in the Euclidean plane, i.e. in the space $(\mathbb{R}^2, d_2)$ where $d_2$ is the usual Euclidean metric. Consider the set $A \subseteq \mathbb{R}^2$ defined as follows: $A$ is the union of all horizontal lines described by equations $y = 1/n$. Equivalently,

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \text{ is arbitrary and } y = 1/n \text{ for some } n \in \mathbb{N}\}.$$

(Draw the picture.)

Determine the closure of $A$. 

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