1. (5pt) Let $F$ be the set of all rational functions $f$ of the form

$$f(x) = \frac{ax + b}{cx + d}$$

with rational coefficients $a, b, c, d$. Show that $F$ is countable.

You can use the following two facts: (1) the set of all rational numbers $\mathbb{Q}$ is countable, and (2) if we have finitely many countable sets $A_1, \ldots, A_n$ then the Cartesian product $A_1 \times \cdots \times A_n$ is countable.

**Sample Solution.** The map $h : \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \to F$ defined by

$$h(\langle a, b, c, d \rangle) = f_{a,b,c,d}$$

where $f_{a,b,c,d} : \mathbb{R} \to \mathbb{R}$ is the rational function defined by

$$f_{a,b,c,d}(x) = \frac{ax + b}{cx + d}$$

is a surjection. We prove this first. This is in fact very easy. If $f$ is a rational function defined by

$$f(x) = \frac{ax + b}{cx + d},$$

then obviously $f = f_{a,b,c,d}$, and so $f = h(\langle a, b, c, d \rangle)$. This proves the surjectivity of $h$.

Since $\mathbb{Q}$ is countable and finite Cartesian products of countable sets are countable, we have $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is countable. Since $h$ is a surjection of $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ onto $F$ and $F$ is obviously infinite, the set $F$ must be countable.

2. (5pt) Given is the set $X$ of all linear functions $f_a : \mathbb{R} \to \mathbb{R}$ of the form $f_a(x) = ax$ where $a \in \mathbb{R}$. Show that the function $d$ defined by

$$d(f_a, f_b) = |a - b|$$

is a metric on $X$.

**Sample solution.** We have to verify that all properties of a metric are satisfied.
First, obviously $d(f_a, f_b) \geq 0$ for all $a, b$. We need to see that $d(f_a, f_b) = 0$ only if $f_a = f_b$. But if $d(f_a, f_b) = 0$ then $a = b$, so $ax = bx$ for all $x$. This means that $f_a = f_b$.

The symmetricity of $d$ is also obvious: $d(f_a, f_b) = |a - b| = |b - a| = d(f_b, f_a)$.

Triangle inequality is easy: $d(f_a, f_c) = |a - c| = |(a - b) + (b - c)| \leq |(a - b)| + |(b - c)| = d(f_a, f_b) + d(f_b, f_c)$.

This completes the proof that $d$ is a metric.

3. Consider the space $BF(\mathbb{R})$ of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ with the sup metric. Let $Z \subseteq BF(\mathbb{R})$ be the set of all functions $f \in BF(\mathbb{R})$ with the property $f(0) = f(1)$.

(a) (2pt) Is $Z$ open?

(b) (3pt) Is $Z$ closed?

Apply the definition of openness and closedness directly (but of course you can use a different approach, if you prefer).

**Sample solution.** (a) $Z$ is not open. To see this, we find some $f \in Z$ such that for every $\varepsilon > 0$ we have $B(f, \varepsilon) \not\subseteq Z$. We let $f$ be the constant function with value 0, i.e. $f(x) = 0$ for all $x \in \mathbb{R}$. Clearly $f \in Z$, since $f$ is bounded (so $f \in BF(\mathbb{R})$) and $f(0) = 0 = f(1)$. Now let $\varepsilon > 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \frac{\varepsilon/2}{1 + x^2}$$

for all $x \in \mathbb{R}$. Clearly $\frac{\varepsilon/2}{1 + x^2} \leq \varepsilon/2$ for all $x \in \mathbb{R}$ since $1 + x^2 \geq 1$. Thus, $g$ is bounded, and so $g \in BF(\mathbb{R})$. Also

$$d(g, f) = \sup_{x \in \mathbb{R}} |g(x) - f(x)| =$$

$$= \sup_{x \in \mathbb{R}} |\frac{\varepsilon/2}{1 + x^2} - 0| =$$

$$= \sup_{x \in \mathbb{R}} \frac{\varepsilon/2}{1 + x^2} \leq$$

$$= \varepsilon/2$$

Since $\varepsilon/2 < \varepsilon$, this tells us that $g \in B(f, \varepsilon)$. But $g \not\in Z$, since $g(0) = \frac{\varepsilon/2}{1 + 0^2} = \varepsilon/2$, whereas $g(1) = \frac{\varepsilon/2}{1 + 1^2} = \varepsilon/4$, and so $g(0) \neq g(1)$. 

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(b) $Z$ is closed. To see this, we show that the complement of $Z$ is open. Assume $f$ is in the complement of $Z$, i.e. $f(0) \neq f(1)$. We find an open ball $B(f, \varepsilon)$ that is completely contained in the complement of $Z$. Assume that $f(0) \neq f(1)$; the case $f(0) > f(1)$ can be treated the same way. Let $\varepsilon = (f(1) - f(0))/2$. We claim that $B(f, \varepsilon)$ is contained in the complement of $Z$. That is, we have to show that $g(0) \neq g(1)$ for all $g \in B(f, \varepsilon)$. Indeed, if $g \in B(f, \varepsilon)$ then

$$g(0) < f(0) + \varepsilon = f(0) + (f(1) - f(0))/2 = (f(1) + f(0))/2 = f(1) - (f(1) - f(0))/2 = f(1) - \varepsilon < g(1).$$

Both the first and the last inequalities follow from the fact that $g \in B(f, \varepsilon)$, which means that $\sup_{x \in \mathbb{R}}|f(x) - g(x)| < \varepsilon$. This completes the argument.

4. (5pt) Work in the Euclidean plane, i.e. in the space $(\mathbb{R}^2, d_2)$ where $d_2$ is the usual Euclidean metric. Consider the set $A \subset \mathbb{R}^2$ defined as follows: $A$ is the union of all horizontal lines described by equations $y = 1/n$. Equivalently,

$$A = \{ \langle x, y \rangle \in \mathbb{R}^2 \mid x \text{ is arbitrary and } y = 1/n \text{ for some } n \in \mathbb{N}, n > 0 \}.$$ (Draw the picture.)

Determine the closure of $A$.

**Sample solution.** Let $L$ be the horizontal line described by the equation $y = 0$. We show that $\overline{A} = A \cup L$. The intuition behind is that we have to add all points that are limits of sequences in $A$.

Obviously $A \subseteq A \cup L$.

Also, $A \cup L$ is closed: If $\langle x, y \rangle \notin A \cup L$, then one of the following possibilities occur:

(a) $y < 0$. In this case let $\varepsilon = |y|$. If $\langle u, v \rangle \in B(\langle x, y \rangle, \varepsilon)$ then $v < 0$, otherwise

$$d(\langle u, v \rangle, \langle x, y \rangle) = \sqrt{(u - x)^2 + (v - y)^2} \geq \sqrt{(v - y)^2} = |v - y| = v + |y| \geq y = \varepsilon,$$

so $\langle u, v \rangle$ would not be in the open ball $B(\langle x, y \rangle, \varepsilon)$. The third equality in the above computation follows from the fact that $y < 0$. So $\langle u, v \rangle \notin A \cup L$. 


(b) $y > 0$, and so we can find the smallest $n$ such that $y \leq \frac{1}{n}$. Such $n$ exists, since the values $\frac{1}{n}$ converge to 0 as $n \to \infty$. We let

$$
\varepsilon = \begin{cases} 
\min \{ y - \frac{1}{n}, \frac{1}{n-1} - y \} & \text{if } n > 1 \\
y - 1 & \text{if } n = 1
\end{cases}
$$

We claim that the ball $B(\langle x, y \rangle, \varepsilon)$ is completely contained in the complement of $A \cup L$. Indeed:

For $\langle u, v \rangle \in B(\langle x, y \rangle, \varepsilon)$ we have

$$
\varepsilon > d(\langle u, v \rangle, \langle x, y \rangle) = \sqrt{(u-x)^2 + (v-y)^2} \geq \sqrt{(v-y)^2} = |v-y|.
$$

(1)

If $n = 1$ then $y > 1$ and (1) gives $|v-y| < \varepsilon = y - 1$. This implies $v > 1$, and thereby $\langle u, v \rangle \notin A \cup L$: If $v \leq 1$ then $|v-y| = y-v \geq y-1$, which is impossible.

If $n > 1$ then $\frac{1}{n} < y < \frac{1}{n-1}$ and (1) gives

$$
|v-y| < \varepsilon = \min \{ y - \frac{1}{n}, \frac{1}{n-1} - y \}.
$$

It follows that $\frac{1}{n} < v < \frac{1}{n-1}$, and hence $\langle u, v \rangle \notin A \cup L$. To see the left inequality: If $v \leq \frac{1}{n}$ then $|v-y| = y-v \geq y - \frac{1}{n}$, so this is impossible.

To see the right inequality: If $v \geq \frac{1}{n-1}$ then $|v-y| = v-y \geq \frac{1}{n-1} - y$, again impossible.

Since (a) or (b) is true for all $\langle u, v \rangle \in B(\langle x, y \rangle, \varepsilon)$, we have that $\langle u, v \rangle \notin A \cup L$ for each such $\langle u, v \rangle$. So we conclude that $B(\langle x, y \rangle, \varepsilon)$ is contained in the complement of $A \cup L$. Hence $A \cup L$ is closed.

The above discussion shows that $\bar{A} \subseteq A \cup L$. It remains to show that every point $\langle x, 0 \rangle$ where $x \in \mathbb{R}$ is in $\bar{A}$. It will follow that $A \cup L \subseteq \bar{A}$, which together with the above inclusion implies $\bar{A} = A \cup L$. But for every $x \in \mathbb{R}$, the pair $\langle x, 0 \rangle$ must be in $\bar{A}$ since the sequence $\langle x, \frac{1}{n} \rangle_n$ converges to $\langle x, 0 \rangle$ and consists of elements of $A$. The limit then $\langle x, 0 \rangle$ then must be in $\bar{A}$. This completes the proof.