MATH 150 HOMEWORK 4

DUE: Friday, November 9
Student Name/Id # (Include all students in the group):

IMPORTANT INSTRUCTION: It is crucial that you clearly state justifications for all your conclusions. This is the point of the homeworks – to practice understanding of the material and the ability to express your understanding.

1. (5pt) Express the following statements in the language of Arithmetic. That is, write down a formula $\varphi(v_1, \dots, v_\ell)$ in the language of Arithmetic such that for all numbers $a_1, \dots, a_\ell \in N$,

$$\mathbb{N} \models \varphi[a_1, \dots, a_\ell]$$

if and only if the property given below is true.

(a) **(1pt)** "w is the greatest common divisor of u, v"
(b) **(1pt)** "w is the smallest common multiple of u, v"
(c) **(1pt)** "There is an arithmetic sequence of length v consisting of primes".
(d) **(1pt)** "There are infinitely many arithmetic sequences of length v consisting of primes".
(e) **(1pt)** "Number u is divisible by four consecutive primes."

Convention: Use abbreviations. So, for instance, if you have a formula $\psi(u)$ expressing "u is a prime", equivalently $\mathbb{N} \models \psi[a]$ iff a is a prime, then write $\psi(u)$ in your formulas to reduce the length. Thus, for instance to say that there are two distinct primes you would write "$(\exists u)(\exists v)(\neg u \doteq v \wedge \psi(u) \wedge \psi(v))$".

2. (5pt) Express the following statements in the language of graphs. That is, write down a formula $\varphi(v_1, \dots, v_\ell)$ in the language of graphs such that if $G = (V, E)$ is a graph and $a_1, \dots, a_\ell \in V$ (so these are vertices),

$$G \models \varphi[a_1, \dots, a_\ell]$$

if and only if the property given below is true. Similarly as in Problem 1, use abbreviations whenever suitable.

Draw pictures that would illustrate problems (a) – (c) below as well as the explanations A – C.

(a) **(1pt)** Given a number n: "The degree of every vertex is at most n". See below for the definition of a degree.
(b) **(1pt)** Given a number n: "The degree of every vertex is precisely n".
(c) **(1pt)** Given an number n: "Any two vertices in V are connected by a path of length at most n." See below for the definition of a path.
(d) **(1pt)** Given a number n: "Any two vertices in V are are on some cycle of length at precisely n." See below for the definition of a cycle.
(e) **(1pt)** Write down a sentence σ in the language of graphs such that for every finite graph G, $G \models \sigma$ if and only if G is a union of disjoint cycles.

Explanations: A. The **degree** of a vertex x in the graph $G = (V, E)$ is the number of edges which have x as one of their end-points, that is, it is the number of edges of the form (x, y) for some y.

B. A **path** in a graph $G = (V, E)$ of length n is a sequence of vertices $x_1, \ldots, x_{n+1}$ such that $x_i E x_{i+1}$ whenever $i < n + 1$. So a path of length n has $n + 1$ vertices and n edges.

C. A **cycle** in a graph of length n is a sequence of distinct vertices $x_1, \ldots, x_n$ such that $x_i E x_{i+1}$ whenever $i < n$ and $x_n E x_1$. So a cycle of length n has n vertices and n edges. However, in the graph G there may be additional edges between x_i, that is, it is allowed that $x_i E x_{i+2}$ for instance. On the other hand, if we demand that the cycle itself is a graph, additional edges are not allowed: A graph $G = (V, E)$ is a cycle of length n if $V = \{x_1, \ldots, x_n\}$ as above, $x_i E x_{i+1}$ whenever $i < n$ and $x_n E x_1$, and G has no other edges.

3. (5pt) Recall the definition of a free occurrence of a variable in a formula. This was defined in the lecture. Also recall the inductive definition of a free occurrence.

(a) **(1pt)** Write down a formula in the language of graphs in which variable u has only free occurrences, variable v has only bound occurrences, and variable w has both free and bound occurrences.

(b) **(2pt)** Give an inductive definition of the number of distinct variables that have free occurrences in a formula φ. This means that you **do not** count multiplicities; i.e. each variable is counted only once.

(c) **(2pt)** In the lecture we had the theorem that the statement "$\mathcal{M} \models \varphi[s]$" depends only on the evaluation of variables with free occurrence in φ. In the discussion you sketched the proof – this went by induction on complexity of φ. Give a rigorous proof of the "quantifier case" in the induction. That is, prove the following:

Assume: ψ is a formula. If s_1, s_2 are evaluations of variables such that $s_1(v) = s_2(v)$ whenever v has a free occurrence in ψ, then

$$\mathcal{M} \models \psi[s_1] \quad \text{if and only if} \quad \mathcal{M} \models \psi[s_2].$$

Prove: Let φ be the formula $(\exists u)\psi$ where ψ is as above. Let $\tilde{s}_1, \tilde{s}_2$ be evaluations of variables such that $\tilde{s}_1(v) = \tilde{s}_2(v)$ whenever v has free occurrence in φ. Then

$$\mathcal{M} \models \varphi[\tilde{s}_1] \quad \text{if and only if} \quad \mathcal{M} \models \varphi[\tilde{s}_2].$$