MATH 150 HOMEWORK 5

DUE: Friday, November 16
Student Name/Id # (Include all students in the group):

IMPORTANT INSTRUCTION: It is crucial that you clearly state justifications for all your conclusions. This is the point of the homeworks – to practice understanding of the material and the ability to express your understanding.

1. (5pt) Work in the language of Arithmetic. Write down a formula expressing the following in the standard model of Arithmetic.

(a) **(1pt)** "u is a sum of three consecutive squares"
(b) **(1pt)** "u is the remainder after dividing v by w"
(c) **(1pt)** "the remainder of v divided by w is a cube"
(d) **(1pt)** "w is a root of infinitely many quadratic polynomials with leading coefficient 1"

Write down a set of formulas expressing the following statement in the standard model of Arithmetic.

(e) **(1pt)** "w is a root of polynomials of arbitrarily large degree with leading coefficient 1."

2. (5pt) Work in the language of graphs. Write down a sentence expressing the following about the graph $G = (V, E)$. Refer to Homework 4 for the definition of a path.

(a) **(1pt)** "Any two vertices in G are connected by at least two distinct paths of length 3."
(b) **(1pt)** "Any vertex of degree 2 is connected by a path of length 3 with a vertex of degree 3."
(c) **(1pt)** "Any two vertices connected by a path of length 3 have degree at most 2."

Write down a set of sentences expressing the following about the graph $G = (V, E)$. This set will be of the form $T = \{\sigma_1, \sigma_2, \dots\}$; where all σ_k follow the same pattern. Write down the general σ_k.

(d) **(1pt)** "There are infinitely many triangles in G."
(e) **(1pt)** "G has arbitrarily large finite induced subgraphs that are trees".

A **triangle** in a graph $G = (V, E)$ is a set of three distinct vertices $\{a, b, c\} \subseteq V$ such that any two vertices from this set are connected by edge. (Draw a picture!) Technically written: $\{a, b\}, \{b, c\}, \{a, c\} \in E$. Notice: if $\dot{E}$ is the binary relational symbol in the language of graphs then the interpretation $\dot{E}^G$ is a binary relation, that is, as subset of $V \times V$. So in this notation we write $(a, b), (b, a), (a, c), (c, a), (b, c), (c, a) \in \dot{E}^G$. Recall also that if $(a, b) \in \dot{E}^G$ then automatically $(b, a) \in \dot{E}^G$, because G is an unoriented graph, so the latter need not be written explicitly. (Similarly for other edges.) So to say that $\{a, b, c\}$ are vertices of a triangle in G it suffices to say for instance that $(a, b), (b, c), (c, a) \in E$.

Refer to the "NOV 3" practice sheet for the definition of a tree. A graph $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E \cap (V \times V)$. Here we use the convention above that views the set of edges as a binary relation. A graph $G' = (V', E')$ is an **induced subgraph** of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' = E \cap (V \times V)$. So the induced subgraph has all edges it possibly can have. Draw the pictures to see the differences!

3. (5pt) Let $\mathcal{L}$ be a language.

(a) **(2pt)** Write down an inductive definition that counts the number of occurrences of the existential quantifier in an $\mathcal{L}$-formula φ.

(b) **(3pt)** Consider variables v and $u_1, \ldots, u_\ell$, and $\mathcal{L}$-formulas $\varphi(v, u_1, \ldots, u_\ell)$ and $\psi(u_1, \ldots, u_\ell)$ having all free variables among the displayed ones. In particular, v is a variable that has **no free occurrence in** ψ. Let s be an evaluation of variables $u_1, \ldots, u_\ell$ in the $\mathcal{L}$-structure $\mathcal{M}$, say $s : u_i \mapsto a_i \in M$ for $i = 1, \ldots, \ell$; here M is the domain of the structure $\mathcal{M}$. Prove: If for every $b \in M$ it is true that

$$\mathcal{M} \models (\varphi \to \psi)[b, a_1, \ldots, a_\ell]$$

where b is an evaluation of variable v, then

$$\mathcal{M} \models \big((\exists v)\varphi \to \psi\big)[a_1, \ldots, a_\ell].$$

Let me emphasize that the "main" connective the formula "$(\exists v)\varphi \to \psi$" is not the existential quantifier, but the implication $\to$.

Also, let me remark that you can approach this problem by direct use of the definition of satisfaction in the straightforward way.