MATH 150 HOMEWORK 6

DUE: Friday, November 30
Student Name/Id # (Include all students in the group):

IMPORTANT INSTRUCTION: It is crucial that you clearly state justifications for all your conclusions. This is the point of the homeworks – to practice understanding of the material and the ability to express your understanding.

1. (5pt) Write down a single formula that expresses the following statements in the given languages.

(a) **(2pt)** $\mathcal{L}$ is the language of arithmetic. Express the statement "number u is a product of one or more consecutive primes". Here we do not specify the number of these primes! To understand better what kind of numbers are considered, here are some examples of such numbers: (i) number $u = 5$ is a product of one prime, namely prime 5. (ii) Number $u = 210$ is a product of four consecutive primes: $210 = 2 \cdot 3 \cdot 5 \cdot 7$. (iii) Number $u = 105$ is a product of three consecutive primes: $105 = 3 \cdot 5 \cdot 7$. (iv) number 1001 is a product of consecutive primes: $1001 = 7 \cdot 11 \cdot 13$.

Hint. You have to say two things: First, that if primes $v < w$ occur in the product, then so does any prime z which is between v and w. Second, that each such prime occurs in the product only once.

(b) **(2pt)** $\mathcal{L}$ is the language of graphs. Given a number $n > 2$, express the statement "vertex v is on two distinct cycles of length n".

(c) **(1pt)** $\mathcal{L}$ is the language of equivalence relations. Express the statement "there are exactly three equivalence classes consisting of one element"

2. (5pt) Write down infinite sets of formulas/sentences expressing the following facts in the given structures.

(a) **(2pt)** $\mathcal{L}$ is the language of graphs. Write down an infinite set of formulas $\{\varphi_0(v), \varphi_1(v), \dots\}$ that express the statement "vertex v has infinite degree".

That is: Given a graph $G = (V, E)$ and a vertex $a \in V$, the following holds: Vertex a has infinite degree if and only if

$$G \models \varphi_n[a]$$

for all $n \in N$.

(b) **(2pt)** $\mathcal{L}$ is the language of graphs. Write down an infinite set of formulas $\{\varphi_0(v, w), \varphi_1(v, w), \dots\}$ that express the statement "there are infinitely many paths of length exactly 3 connecting v with w". Here the meaning of "express" is analogous to that in (a).

(c) **(1pt)** $\mathcal{L}$ is the language of equivalence relations. Write down an infinite set of formulas $\{\varphi_0(v), \varphi_1(v), \dots\}$ that express the statement "the equivalence class of v is infinite". Again, the meaning of "express" is analogous to that in (a).

3. (5pt) Given is a language $\mathcal{L}$. Review the following notions from the lecture; see also the text "Basic notions ..." on the course webpage.

- "$\mathcal{M} \models \Sigma$" where Σ is a set of $\mathcal{L}$-sentences.
- "$\Sigma \models \sigma$" Recall that if Σ consists of a single sentence τ, i.e. if $\Sigma = \{\tau\}$, we write "$\tau \models \sigma$" in place of "$\{\tau\} \models \sigma$".

Work on the following tasks.

(a) **(1+1pt)** Assume τ, τ' and σ are $\mathcal{L}$-sentences. Prove:

$$\tau \vee \tau' \models \sigma \quad \text{if and only if} \quad \tau \models \sigma \text{ and } \tau' \models \sigma.$$

(b) **(1pt)** In this task $\mathcal{L}$ is the language of equivalence relations and ψ_1, ψ_2, ψ_3 are the axioms of equivalence relations. Given a number $n > 0$, let φ_n be a sentence that expresses the statement "there is at least one equivalence class of size n or more". Consider the set of sentences Σ defined as follows:

$$\Sigma = \{\psi_1, \psi_2, \psi_3\} \cup \{\varphi_n \mid n = 1, 2, \ldots\}.$$

Prove/Disprove the following statement. For every $\mathcal{L}$-structure $\mathcal{M}$:

If $\mathcal{M} \models \Sigma$ then $\mathcal{M}$ has an infinite equivalence class.

(c) **(1+1pt)** Consider two sets of $\mathcal{L}$-sentences Σ, Σ' such that $\Sigma' \subseteq \Sigma$, and an $\mathcal{L}$-sentence σ. Prove/disprove:

(i) If $\Sigma' \models \sigma$ then $\Sigma \models \sigma$.

(ii) If $\Sigma \models \sigma$ then $\Sigma' \models \sigma$.

Remark. In order to disprove incorrect statements, it is often sufficient to find a counterexample. But in each case you have to explain why this is sufficient.