MATH 150 FALL 2012
PRACTICE PROBLEMS NOV 5

1. Work in the language $\mathcal{L}$ of equivalence relations. Write down a sentence σ that describes the following properties of the equivalence relation E.

(a) E has at most five equivalence classes.
(b) E has at least five equivalence classes.
(c) E has precisely three equivalence classes.
(d) E had precisely two equivalence classes and each of them has precisely three elements.

Here "σ" describes the property (a)" means that for every $\mathcal{L}$-structure $\mathcal{M} = (M, E)$ where E is an equivalence relation on $\mathcal{M}$,

$$\mathcal{M} \models \sigma \quad \text{if and only if} \quad E \text{ has at most five equivalence classes.}$$

Similarly for (b), (c) and (d).

2. Work in the language $\mathcal{L}$ of equivalence relations. Write down a set of sentences Σ that describes the following properties of the equivalence relation E. Give the precise meaning to the term "Σ describes the property", following the explanation in Problem 1.

(a) E has infinitely many equivalence classes.
(b) E has precisely two equivalence classes and at least one of them is infinite.
(c) E has precisely two equivalence classes and both are infinite.
(d) For each $n > 0$ the equivalence relation E has an equivalence class with precisely n elements.

3. Work in the language of partial orderings $\mathcal{L} = \{\dot{\leq}\}$. Let $\mathcal{N}^+ = (N^+, |)$ be the following structure for $\mathcal{L}$: N^+ is the set of all positive integers, and $|$ is the divisibility relation. This means that we write $a \,|\, b$ to express that number a divides number b.

(a) Prove that the axioms of partial orderings hold in $\mathcal{N}^+$ where the symbol cot$\leq$ is interpreted as $|$, that is, $\dot{\leq}^{\mathcal{N}^+}$ is the relation $|$.

Write down the following statements in the language of partial orderings.

(b) "w is the greatest common divisor of u, v".
(c) "w is the smallest common multiple of u, v".
(d) "u is a prime".
(e) "u, v are relatively prime".

4. Work in the language of partial orderings $\mathcal{L} = \{\dot{\leq}\}$. Let $\mathcal{A} = (\mathbb{A}, \subseteq)$ be the following structure for $\mathcal{L}$: $\mathbb{A}$ is the power set of a set A given in advance, that is,

$$\mathbb{A} = \text{the set of all sets } x \text{ such that } x \subseteq A,$$

and $\subseteq$ is the inclusion of sets. So $x \subseteq y$ if and only if all elements of x are elements of y.

(a) Prove that the axioms of partial orderings hold in $\mathcal{A}$ where $\dot{\leq}$ is interpreted as the inclusion relation, that is, $\dot{\leq}^{\mathcal{A}}$ is the relation $\subseteq$.

Write down the following statements in the language of partial orderings.

(b) “w is the intersection of u, v”.
(c) “w is the union of u, v”.
(d) “u is the empty set.”
(e) “u is the set A.”
(f) “u, v have nonempty intersection”.
(g) “v is the complement of u.”
(h) “u is a singleton”.
(i) “u has precisely two elements.”

Compare (b,c) of Problem 4 with (b,c) of Problem 3!

5. Work in the language of partial orderings $\mathcal{L} = \{\dot{\leq}\}$. Let A be the usual alphabet $a, b, c, d, e, \dots$ and $\mathbb{A}$ be the set of all finite strings (also called “words”) that are built from letters in alphabet A. You can also think about strings as finite sequences of symbols coming from A. The order here is important, that is, it is important what letter is the first one, what letter is the second one, etc., like in words in our human language. We also consider the **empty** string, i.e. the string containing no symbols. The **length** of a string is the number of letters occurring in the string, where we count multiplicities. So for instance the length of the strings “aba”, “abc”, “aaa” is 3; also the length of the empty string is 0. Let $\mathcal{A} = (\mathbb{A}, \trianglelefteq)$ be the structure for $\mathcal{L}$ where $\trianglelefteq$ is the relation “initial segment”. So given two words $w_1, w_2 \in \mathbb{A}$, $w_1 \trianglelefteq w_2$ means that w_1 is an initial segment of w_2 where we understand that the initial segment need not be necessarily proper, that is, it is allowed that $w_1 = w_2$.

(a) Prove that the axioms of partial orderings hold in $\mathcal{A}$ where $\dot{\leq}$ is interpreted as the relation $\trianglelefteq$, that is, $\dot{\leq}^{\mathcal{A}}$ is the relation $\trianglelefteq$.

Write down the following statements in the language of partial orderings.

(b) “w is a common initial segment of u, v”.
(c) “w is the largest common initial segment of u, v”.
(d) “w is the empty string”.
(e) “w is a string of length 1”.
(f) “w is a string of length 2”.
(g) “w is an initial segment of w' shorter by one symbol.
(h) “w is an initial segment of w' shorter by two symbols.

6. Work in the language of strict linear orderings with additional unary relational symbol $\dot{A}$; so $\mathcal{L} = \{\dot{<}, \dot{A}\}$. Consider the $\mathcal{L}$ structure structure $\mathcal{Q} = (\mathbb{Q}, <, A)$ where $\mathbb{Q}$ is the set of all rational numbers, $<$ is the usual ordering of rational numbers, and $A \subseteq \mathbb{Q}$ is a set of rational numbers. The interpretations of the $\mathcal{L}$ symbols in $\mathcal{Q}$ are thus as follows: $\dot{<}^{\mathcal{Q}}$ is the relation $<$, and $\dot{A}^{\mathcal{Q}}$ is the set A.

We say that a set A is **dense** in $\mathbb{Q}$ if and only if for any $p, q \in \mathbb{Q}$ such that $p < q$ there is some $a \in A$ such that $p < a < q$. A number $p \in \mathbb{Q}$ is an **upper bound** on the set A if and only if $a < p$ for all $a \in A$. The notion of **lower bound** on A is defined similarly: define it on your own! We say that a number $p \in \mathbb{Q}$ is a **limit point** of A if and only if every open interval (q, q') that contains p has nonempty intersection with A. Write down statements in the language $\mathcal{L}$ that express the following facts.

(a) “A is a dense subset of $\mathbb{Q}$.”
(b) “A has no upper bound in $\mathbb{Q}$”. (We also say that A is **unbounded** in $\mathbb{Q}$.)
(c) “u is a limit point of A”.

(d) "w is the limit point of the set of all limit points of A".
(e) "w is the only limit point of A".
(f) "A has no limit points".
(g) "A has precisely one limit point."
(h) "A has precisely two limit points."
(i) "the set of all limit points of A is dense in $\mathbb{Q}$".
(j) "the set of all limit points of A is unbounded in $\mathbb{Q}$".