HOMEWORK 4

Due: Wednesday November 4, 2009

Important note: You can quote any result proved in the lecture. I recommend to do so, so that you can focus on each Problem itself.

1. Let \((A, <)\) be a well-ordering and \(f : A \rightarrow A\) be an order-preserving function, i.e. for \(a, b \in A\) we have \(a < b \implies f(a) < f(b)\). Show that \(f(a) \geq a\) for all \(a \in A\).
   
   Hint. Assuming the contrary, build an infinite descending chain of elements of \(A\).

2. Show that if \((A, <)\) is a well-ordering then it is not isomorphic to any of its proper initial segments.
   
   Hint. Apply Problem 1.

3. Prove that for all ordinals \(\alpha, \beta\) we have
   
   \[
   \alpha + \beta = \text{otp} \left( \{0\} \times \alpha \cup \{1\} \times \beta, <_{\text{lex}} \right)
   \]
   
   Hint. Verify that the right side satisfies the recursion condition in the definition of \(\alpha + \beta\).

4. Use the result from Problem 3 to prove that the operation \(+\) on ordinals is associative.
   
   Hint. This amounts to showing that two well-orderings are isomorphic.