HOMEWORK 6

Due: Wednesday November 25, 2009

Important note: You can quote any result proved in the lecture. I recommend to do so, so that you can focus on each Problem itself.

1. Let $A$ be a set. Assuming that there is a selector on $\mathcal{P}(A)$, that $A$ can be well-ordered.

   **Hint.** Fix a selector $F$ on $\mathcal{P}(A)$ and some $s \notin A$. Using recursion on ordinals define a function $g : \text{On} \to A \cup \{s\}$ such that $g(\alpha) = F(A - g[\alpha])$ whenever $A - g[\alpha] \neq \emptyset$. Argue that for a suitably chosen $\alpha^* \in \text{On}$ we have a bijection $g \upharpoonright \alpha^* \to A$. Use this bijection to define a well-ordering on $A$. The construction of $g$ is similar to the “modern” proof of Zorn’s Lemma presented in the lecture and to the proof of the theorem that every well-ordering is isomorphic to a well-ordered set, also presented in the lecture.

2. Let $\alpha$ be a limit ordinal.

   (a) Let $g : \delta \to \alpha$ be a normal function cofinal in $\alpha$ where $\delta$ is an ordinal. Show that $\text{rng}(g)$ is a closed unbounded subset of $\alpha$.

   (b) Let $C \subseteq \alpha$ be a closed unbounded subset of $\alpha$. Show that there is an ordinal $\delta \leq \alpha$ and a normal function $g : \delta \to \alpha$ such that $\text{rng}(g) = C$.

   **Hint.** Regarding (b) $\implies$ (a), consider the unique isomorphism between $\text{otp}(C)$ and $C$. 

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