HOMEWORK 3

Due: Wednesday January 27, 2009

Important note: You can quote any result proved in the lecture. I recommend to do so, so that you can focus on each Problem itself.

1. Work in ZF. Let $\kappa$ be an infinite cardinal. We define a class $H_\kappa$ as follows

$$H_\kappa = \text{the class of all sets } x \text{ s.t. } \text{card}(\text{trcl}(\{x\})) < \kappa.$$ 

Recall that $\text{trcl}(z)$ is the transitive closure of $z$. Follow the steps below to prove that $H_\kappa$ is a set.

(a) Show that if $x, y$ are sets then

$$x = y \iff \text{trcl}(\{x\}) = \text{trcl}(\{y\}).$$

The direction from left to right is of course trivial. You will need to use the Axiom of Foundation here for the direction from right to left.

Notice that we need to consider $\text{trcl}(\{x\})$ here rather than $\text{trcl}(x)$. To see why, give an example of sets $x \neq y$ such that $\text{trcl}(x) = \text{trcl}(y)$.

The point of this statement is that it enables to say what $x$ is once we know what $\text{trcl}(\{x\})$ is. Write down a formula in the language of set theory which defines $x$ from $\text{trcl}(\{x\})$.

(b) Granting that $\text{trcl}(\{x\})$ is of cardinality smaller than $\kappa$, there is some $\mu < \kappa$ and some bijection $f: \mu \to \text{trcl}(\{x\})$. Use this bijection to define a binary relation $E \subseteq \mu \times \mu$ such that the structures $(\mu, E)$ and $(\text{trcl}(\{x\}), \in)$ are isomorphic. Notice that $E$ is extensional and well-founded.

(c) Define a function $G: \kappa \times \mathcal{P}(\kappa \times \kappa) \to \text{V}$ as follows:

$$G(\alpha, A) = \begin{cases} x & \text{if } A \subseteq \alpha \times \alpha \text{ and } (\alpha, A) \text{ is isomorphic to } (\text{trcl}(\{x\}), \in) \\ \emptyset & \text{otherwise.} \end{cases}$$

Prove that $G$ is a well-defined class function (you don’t have to write a formula formally, but give an informal explanation) and $\text{rng}(G) = H_\kappa$.

Conclude that $H_\kappa$ is a set.
The structure $H_κ$ is of fundamental importance in set theory and we will study it in more detail. For the moment, we just conclude the following.

2. Prove that $H_κ \subseteq V_κ$.
   **Hint.** Use Homework 1, Problem 1(b).

3. Let $I \neq \emptyset$ and $κ_ι \leq λ_ι$ be cardinals such that $λ_ι > 1$ for all $ι$. Show that

   \[ \sum_{ι \in I} κ_ι \leq \prod_{ι \in I} λ_ι. \]

   **Hint.** Notice that you may w.l.o.g. assume that $κ_ι > 0$. Treat the cases $\text{card}(I) = 1, 2$ separately. If $\text{card}(I) > 2$, construct an injection

   \[ F : \bigcup_{ι \in I} (\{ι\} \times κ_ι) \rightarrow \prod_{ι \in I} λ_ι. \]