HOMEWORK 5

Due: Friday March 5, 2009

Important note: You can quote any result proved in the lecture. I recommend to do so, so that you can focus on each Problem itself.

1. Prove the following. Notice the analogy between On and inaccessible cardinals, and also between “proper class” and “unbounded in $\kappa$”.
   (a) The class of all cardinals $\mu$ such that $\mu = \aleph_\mu$ is a closed proper class.
   (b) The class of all strong limit cardinals is a closed proper class.
   (c) Let $\kappa$ be weakly inaccessible. The set of all cardinals $\mu < \kappa$ such that $\mu = \aleph_\mu$ is a closed unbounded subset of $\kappa$.
   (d) Let $\kappa$ be strongly inaccessible. The set of all strong limit cardinals $\mu < \kappa$ is a closed unbounded subset of $\kappa$.

Hint. Use the standard closure argument.

2. Given a cardinal $\kappa$, let
   
   $$ R_\kappa = \{ \mu < \kappa \mid \mu \text{ is regular} \}. $$

   (a) A weakly inaccessible cardinal $\kappa$ is weakly Mahlo iff $R_\kappa$ is stationary in $\kappa$. Prove that if $\kappa$ is weakly Mahlo then

   $$ WI_\kappa = \{ \mu < \kappa \mid \mu \text{ is weakly inaccessible} \} $$

   is stationary in $\kappa$.

   (b) A strongly inaccessible cardinal $\kappa$ is Mahlo iff $R_\kappa$ is stationary in $\kappa$. Prove that if $\kappa$ is Mahlo then

   $$ I_\kappa = \{ \mu < \kappa \mid \mu \text{ is strongly inaccessible} \} $$

   is stationary in $\kappa$.  

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**Hint.** For (a) use Problem 1(c); for (b) use Problem 1(d).

3. Let $\kappa$ be a regular cardinal and $S \subseteq \kappa$ be a stationary subset of $\kappa$. An ordinal $\delta < \kappa$ is called a **reflection point** of $S$ iff $\text{cf}(\delta) > \omega$ and the set $S \cap \alpha$ is stationary in $\alpha$. Prove that the set

$$\text{NON}_\kappa = \{ \delta \in S \mid \delta \text{ is not a reflection point of } S \}$$

is a stationary subset of $\kappa$. (Intuitively, $S$ contains many non-reflection points of itself.)

**Hint.** Given a closed unbounded set $C \subseteq \kappa$ in $\kappa$, let $\text{lim}(C)$ be the set of all limit points of $C$. Look at the least ordinal in $S \cap \text{lim}(C)$ (why is the intersection nonempty?). Discuss the two cases where the cofinality of this ordinal is $\omega$ and $\omega_1$.  

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