HOMEWORK 1

Due: Monday April 26, 2010

Important note: You can quote any result proved in the lecture. I recommend to do so, so that you can focus on each Problem itself.

1. Let $\mathcal{L}$ be the language of set theory $\{\in\}$. We will write $\in$ instead of $\in^*$ which involves an abuse of notation as $\in$ is the membership relation itself.

   (a) A **bounded** existential quantification is a quantification of the form
   $$(\exists z)(z \in x \land \varphi).$$
   We abbreviate the above formula by $(\exists z \in x)\varphi$ and call the expression
   “$(\exists z \in x)$” a **bounded existential quantifier**.

   (b) A **bounded** universal quantification is a quantification of the form
   $$(\forall z)(z \in x \rightarrow \varphi).$$
   We abbreviate the above formula by $(\forall z \in x)\varphi$ and call the expression
   “$(\forall z \in x)$” a **bounded universal quantifier**.

   (c) The usual quantifiers $(\exists z)$ and $(\forall z)$ are then called **unbounded**.

   (d) A formula in the language of set theory is **bounded** if and only if it does not contain any unbounded quantifiers. Alternatively, the notion of a bounded formula can be defined inductively: (i) all atomic formulae are bounded, (ii) if $\varphi$ and $\psi$ are bounded then so is $\varphi \land \psi$, (iii) if $\varphi$ is bounded then so is $\neg \varphi$, and (iv) if $\varphi$ is bounded then so are $(\exists z \in x)\varphi$ and $(\forall z \in x)\varphi$.

   Let $M$ be a transitive set. Then $M = \langle M, \in \cap (M \times M) \rangle$ is an $\mathcal{L}$-structure. Here $\in \cap M \times M = \{\langle a, b \rangle \in M \mid a \in b\}$. Prove the following:

   If $\varphi(x_1, \ldots, x_\ell)$ is a bounded formula with all free variables among $x_1, \ldots, x_\ell$ and $a_1, \ldots, a_\ell \in M$ then $M \models \varphi[a_1, \ldots, a_\ell]$ if and only if $\varphi[a_1, \ldots, a_\ell]$ holds in $V$.

   Notice that the problem is not entirely correctly formulated. Try to suggest a correct formulation, but do not run the proof using this correct formulation, as it would add some notational complexity.

   **Hint.** Do the induction on complexity of formulae.

2. Let $\mathcal{L}$ be any language. Working in $ZF$, this exercise leads to a proof of an analogue of the theorem on Henkin models which avoids the notion of provability. This requires suitable substitutes for the notions of consistency, completeness and Henkin constant. Let $\Sigma$ be a set of $\mathcal{L}$-sentences.
For the purpose of this exercise, let us say that:

- $\Sigma$ has Henkin constants in the semantical sense just in case that for every $\mathcal{L}$-formula $\varphi$ with the only free variable $x$ such that $\Sigma \models (\exists x)\varphi$ there is a constant symbol $c$ of the language $\mathcal{L}$ such that $\Sigma \models \varphi(x/c)$.

- $\Sigma$ is complete in the semantical sense if and only if for every $\mathcal{L}$-sentence $\sigma$ it is the case that $\Sigma \models \sigma$ or $\Sigma \models \lnot \sigma$.

Assume that $\Sigma$ is a set of $\mathcal{L}$-sentences such that

(a) Every finite $\Delta \subseteq \Sigma$ has a model;

(b) $\Sigma$ is complete in the semantical sense;

(c) $\Sigma$ has Henkin constants in the semantical sense.

Prove that $\Sigma$ has a model $\mathcal{M}$ such that there is a surjection of $C^\mathcal{L}$ onto the domain of $\mathcal{M}$; here $C^\mathcal{L}$ is the set of all constant symbols of $\mathcal{L}$.

**Hint.** Imitate the proof of Theorem 3.11 from the lecture.