MATH 280A FALL 2018 HOMEWORK 2

Target date: Tuesday, November 6

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

Recall that if α, β are ordinals we often write "$\alpha < \beta$" for "$\alpha \in \beta$".

1. (5+5 lines) Prove the following, but do **not** use ordinals in the proof of (a). In the proof of (b), refer to (a) whenever possible.

(a) Let $(A, <_A)$ be a well-ordered set. If $f : A \to A$ is the $<_A$-preserving map (i.e. $x <_A y \implies f(x) <_A f(y)$) then $a \leq_A f(a)$ for every $a \in A$.

(b) If α, β are ordinals and $f : \alpha \to \beta$ is an order-preserving map then $\alpha \leq \beta$ and $\xi \leq f(\xi)$ whenever $\xi < \alpha$.

2. (7+7 lines) Prove the following.

(a) If α is an ordinal then $S(\alpha) = \alpha \cup \{\alpha\}$ is the smallest ordinal larger than α.

(b) If A is a set of ordinals then $\beta = \bigcup A$ is the smallest ordinal β with the property $\alpha \leq \beta$ for all $\alpha \in A$.

3. (1/2 + 1/2 page) Assume $(A, <_A)$ and $(B, <_B)$ are well-ordered sets. Recall the definitions of the lexicographic ordering $<_{\text{lex}}$ on $A \times B$ and the maximo-lexicographic ordering $<_{\text{mlex}}$ on $A \times A$:

$$(a,b) <_{\text{lex}} (a',b') \iff (a <_A a' \text{ or } (a = a' \text{ and } b <_B b'))$$

$$(a,b) <_{\text{mlex}} (a',b') \iff \begin{cases} \max\{a,b\} < \max\{a',b'\} \text{ or} \\ \max\{a,b\} = \max\{a',b'\} \text{ and } (a,b) <_{\text{lex}} (a',b') \end{cases}$$

(i) Prove that $<_{\text{lex}}$ is a well-ordering on $A \times B$.

(ii) Prove that $<_{\text{mlex}}$ is a well-ordering on $A \times A$.

4. (3 x 1/2 page) We way that a set a is finite iff there is a bijection $f : n \to a$ for some $n \in \omega$. This corresponds to the intuitive notion of finiteness. Also a finite sequence of elements of a class A is a function $s : n \to A$ for for some $n \in \omega$. Again, this corresponds to the intuitive notion. If $s(i) = a_i$ for $i < n$, we informally often write $(a_0, \ldots, a_{n-1})$ or $\langle a_0, \ldots, a_{n-1} \rangle$ for s. The class of all finite subsets of A is denoted by $[A]^{<\omega}$; the class of all finite sequences of A is denoted by ${}^{<\omega}A$. If A is a set, one can prove that both $[A]^{<\omega}$ and ${}^{<\omega}A$ are sets.

(a) We have seen that the natural well-ordering $< = \in$ on $\mathbf{On}$ is set-like. Is the lexicographical ordering on $\mathbf{On} \times \mathbf{On}$ set-like? Is it a well-ordering? Answer the two questions for maximo-lexicographical ordering as well.

(b) Define a relation $<^*$ on $[\mathbf{On}]^{<\omega}$ by

$$a <^* b \iff \max(a \triangle b) \in b$$

Here $a \triangle b$ is the symmetric difference of a, b which is defined by

$$a \triangle b = (a \smallsetminus b) \cup (b \smallsetminus a)$$

Prove that $<^*$ is a set-like well-ordering on $[\mathbf{On}]^{<\omega}$.

(c) Let $(A, <_A)$ be a linear ordering. We can define $<^*_A$ on $[A]^{<\omega}$ the same way as in (b). Also let $D(A, < \omega)$ be the class of all strictly descending finite sequences of elements of A, that is,

$$D(A, < \omega) = \{s \in {}^{<\omega}A \mid s \text{ is strictly decreasing.}\}$$

Define the lexicographical ordering on $D(A, < \omega)$ by

$$s <_{\mathrm{lex}} t \iff s(i_{s,t}) <_A t(i_{s,t})$$

where

$$i_{s,t} = \text{the least } i \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \text{ such that } s(i) \neq t(i)$$

(so it is assumed that $i_{s,t}$ must exist whenever $s <_{\mathrm{lex}} t$). Find a natural bijection between $D(A, < \omega)$ and $[A]^{<\omega}$ which is an isomorphism between $(D(A, < \omega), <_{\mathrm{lex}})$ and $([A]^{<\omega}, <^*_A)$.

5. (5 lines +1/2 page) Let A be a set and $\mathcal{F}$ be a chain of well-orderings on subsets of A. That is,

(i) elements of $\mathcal{F}$ are well-orderings $(a, <_a)$ where $a \subseteq A$, and

(ii) if $(a, <_a), (b, <_b) \in \mathcal{F}$ then either $a \subseteq b$ and $<_a \subseteq <_b$ or vice versa.

Assume

$$\bigcup \{a \in \mathcal{P}(A) \mid (a, <_a) \in \mathcal{F}\} = A,$$

and let

$$< = \bigcup \{<_a \mid (a, <_a) \in \mathcal{F}\}$$

(you may want to write this more rigorously). Decide the following.

(a) Is $<$ a linear ordering on A? Is $<$ a well-ordering on A?

(b) What is the answer to the second question in (a) if we assume that for any two $(a, <_a), (b, <_b) \in \mathcal{F}$, one of them is an end-extension of the other?