MATH 280A FALL 2018 HOMEWORK 3

Target date: Tuesday, November 20

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do <u>not</u> re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

Recall that if α, β are ordinals we often write "$\alpha < \beta$" for "$\alpha \in \beta$".

1. (5 + 10 lines) Let R be a binary relation on a set A and $f : A \to \mathbf{On}$.

(a) Assume
$$aRb \implies f(a) < f(b)$$
whenever $a, b \in A$. Prove that R is well-founded.

(b) Prove that if f is as in (a) then $\mathsf{rank}_R(a) \leq f(a)$ for every $a \in A$.

2. (1/2 page) Prove that if $a \in \mathsf{WF}$ then $\mathsf{rank}(a)$ is the least ordinal α such that $a \subseteq V_\alpha$.

3. (1/2 + 1/2 page) Let α and β be ordinals. Prove the following.

(i) $\alpha + \beta = \mathsf{otp}((\{0\} \times \alpha) \cup (\{1\} \times \beta), <_{\text{lex}})$.

(ii) $\alpha \cdot \beta = \mathsf{otp}(\beta \times \alpha, <_{\text{lex}})$.

4. (1/2 page + 1/4 page) Assume α is an ordinal or $\alpha = \mathbf{On}$. If $f : \alpha \to \alpha$ is a function (in the latter case we mean a class function), we way that ξ is a **fixpoint** of f iff $f(\xi) = \xi$. Denote the set/class of all fixpoints of f by C_f.

(i) Assume α is an ordinal of uncountable cofinality. Prove that for every normal function $f : \alpha \to \alpha$, the set C_f is a closed (in the interval topology) and unbounded subset of α. Does the conclusion hold for α of countable cofinality?

(ii) Prove that for every normal class function $f : \mathbf{On} \to \mathbf{On}$, the class C_f is a closed proper class. (Just explain how to run the argument from (i) for proper classes and give details only for the points which are new here.) Conclude that the class of all cardinals κ such that $\kappa = \aleph_\kappa$ is a closed proper class.

5. (2/3 page) Let κ be an infinite cardinal. Prove that the following are equivalent.

(i) κ is singular.

(ii) There is a $\gamma < \kappa$ and a sequence of sets $(A_\xi \mid \xi < \gamma)$ such that $\kappa = \bigcup_{\xi<\kappa} A_\xi$ and $\mathsf{card}(A_\xi) < \kappa$ for all $\xi < \gamma$.

Try to understand what (ii) says.

6. (5 lines + 2/3 page + 5 lines) Recall that $\mathbb{Q}$ denotes the set of all rational numbers and $\mathbb{R}$ denotes the set of all real numbers. We consider the natural ordering and the natural interval topology in both cases.

(i) Let $\alpha \geq \omega$ be a countable ordinal. Determine $\mathsf{cf}(\alpha)$.

(ii) Prove by induction on α that for every $\alpha < \omega_1$ there is a function $f : \alpha \to \mathbb{Q}$ which is strictly increasing and continuous.

(ii) Prove that there is no strictly increasing function $f : \omega_1 \to \mathbb{R}$.