MATH 280A FALL 2018 HOMEWORK 4

Target date: Friday, December 14

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

Recall that if α, β are ordinals we often write "$\alpha < \beta$" for "$\alpha \in \beta$".

1. (1/2 page) Work in ZFC. Given cardinals κ, μ we denote

$$[\kappa]^\mu = \{x \in \mathcal{P}(\kappa) \mid \mathsf{card}(x) = \mu\}$$

Prove that if $\mu \leq \kappa$ are cardinals and κ is infinite then

$$[\kappa]^\mu \sim {}^\mu\kappa$$

where $\sim$ denotes the relation equinumerosity. One "half" of the relation $\sim$ can be proved in ZF which one is it?

2. (3 lines) Work in ZFC. Prove that if κ is an infinite cardinal then $\kappa^\kappa = 2^\kappa$.

3. (10 lines) Work in ZFC. Let μ be a cardinal and $(\kappa_\xi \mid \xi < \mu)$ be a strictly increasing sequence of cardinals. Denote

$$\kappa = \sup_{\xi<\mu} \kappa_\xi$$

Prove

$$\prod_{\xi<\mu} \kappa_\xi = \kappa^\mu$$

4. (1 page) Work in ZF. Let κ be an infinite cardinal. Denote

$$H_\kappa = \{x \mid \mathsf{card}(\mathsf{trcl}(x)) < \kappa\}.$$

Prove that H_κ is a transitive set.

5. (1/2 page) Work in ZF. Let M be a nonempty transitive set. We can view $(M, \in)$ as a structure for the language of set theory (LST). Prove that the structure $(M, \in)$ satisfies axioms of existence, extensionality and foundation.

6. (1 page + 1/3 page + 5 lines) Let $\alpha > \omega$ be a limit ordinal. Consider $(V_\alpha, \in)$ as a structure for LST.

(a) Prove that if σ is any axiom of ZF other than an axiom belonging to Collection Schema then $(V_\alpha, \in) \models \sigma$. Refer to Problem 5 for the first three axioms. Also prove that AC implies that $(V_\alpha, \in)$ satisfies AC. Conclude that axioms of Collection Schema do not follow from the rest of the axioms of ZF or even ZFC.

(b) Prove that if α is a strongly inaccessible cardinal then $(V_\alpha, \in)$ satisfies all axioms of ZF.

(c) By choosing α suitably, prove that Collection Schema is necessary for the conclusion that cardinals constitute a proper class.

7. (1 page + 1/2 page + 5 lines) Work in ZF. Let $\kappa > \omega$ be a cardinal. Consider $(H_\kappa, \in)$ as a structure for LST.

(a) Prove that if σ is any axiom of ZF other than an axiom belonging to Collection Schema or the Power axiom then $(V_\alpha, \in) \models \sigma$. Refer to Problem 5 for the first three axioms. Also prove that AC implies that AC holds in $(H_\kappa, \in)$.

(b) Prove that if κ is regular and σ is any axiom of ZF other than the power set axiom then $(H_\kappa, \in) \models \sigma$. Working in ZFC, conclude that the Power Set axiom does not follow from the rest of the axioms of ZF or even ZFC, and also that the Power Set Axiom is necessary for the conclusion that cardinals constitute a proper class.

(c) Prove that if κ is a strongly inaccessible cardinal then $(H_\kappa, \in)$ satisfies all axioms of ZF.

Optional problems

8. This exercise concerns an application of the compactness theorem. Let $\mathcal{L} = \{E\}$ be the language of graphs.

A **loop** of length $n \in \omega \setminus \{0\}$ in a graph G is a finite sequence of nodes $(a_0, a_1, \ldots, a_n)$ such that $(a_i, a_{i+1}) \in E^G$ for all $i < n$ and $(a_n, a_0) \in E^G$. A graph G is a **tree** iff G does not have any finite loop.

(a) Prove that there exists a set of $\mathcal{L}$-sentences Σ such that for any graph G we have

$$G \models \Sigma \iff G \text{ is a tree}$$

(b) Prove that there does not exist a set of $\mathcal{L}$-sentences Γ such that for any graph G we have

$$G \models \Gamma \iff G \text{ is not a tree}$$

9. This exercise provides a connection between the notions of compactness from first-order logic and that from topology. Let $\mathcal{L}$ be a language and M be a nonempty set. Let X be the set of all $\mathcal{L}$-structures with domain M. Notice that this is indeed a set. For every $\mathcal{L}$-sentence σ, let

$$B_\sigma = \{\mathcal{M} \in X \mid \mathcal{M} \models \sigma\}$$

(a) Prove that the collection

$$\mathcal{B} = \{B_\sigma \mid \sigma \text{ is an } \mathcal{L}\text{-sentence}\}$$

constitutes a basis of a topology on X; denote this topology by $\mathcal{T}_\mathcal{L}$. Show that the sets B_σ are clopen in this topology.

(b) Prove that the topological space $(X, \mathcal{T}_\mathcal{L})$ is a compact Hausdorff topological space.