HOMEWORK 2

1. Prove that $\in$ is a strict linear ordering on ω.

Hint. Transitivity and irreflexivity of $\in$ are easy consequences of the facts we proved in the lecture. Transitivity follows since every element of ω is a transitive set. Irreflexivity follows from the well-foundedness of $\in$ on ω. Write down **how** the transitivity and irreflexivity can be derived from those two facts.

The only part that requires some work is the trichotomicity of $\in$. You need to show that for every $x, y \in \omega$ either $x \in y$ or $x = y$ or $y \in x$. This is done by simultaneous induction. Say prove this by induction on y. The base step $y = \varnothing$ has to be proved by induction on x, and is straightforward. The induction step in the induction on y is straightforward up to one point: you need to verify that $y \in x \longrightarrow (S(y) \in x \vee S(y) = x)$. Do this by induction on x.

2. Study the properties of transitive closures.

(a) Find a pair of sets x, y such that $x \neq y$ but $\operatorname{trcl}(x) = \operatorname{trcl}(y)$.

(b) Work in ZF, that is, also consider the Axiom of Foundation. Show that if x, y are sets and $\operatorname{trcl}(\{x\}) = \operatorname{trcl}(\{y\})$ then $x = y$.

Hint. Assuming $\operatorname{trcl}(\{x\}) = \operatorname{trcl}(\{y\})$ and $x \neq y$ show that $x \in \operatorname{trcl}(x)$; from this find a finite set with no $\in$-minimal element.

3. A **sequence of elements of A of length n** is a function $s : n \to A$ where $n \in \omega$. The class of all such sentences is denoted by ${}^{n}A$. We let

$$^{<\omega}A = \bigcup_{n \in \omega} {}^{n}A = \{s \,|\, (\exists n \in \omega)(s : n \to A)\}.$$

The elements of $^{<\omega}A$ are called **finite sequences of elements of A**.

Prove that if A is a set then also $^{<\omega}A$ is a set.

Hint. This is a proof by recursion. Design a function $G : \mathbf{V} \to \mathbf{V}$ that will determine the induction step. Notice: you have to prove, using the axioms, that G is a class function. This includes proving that the values of G are sets.