HOMEWORK 3

1. In the proof of the theorem on construction by recursion (Theorem 1.26) we defined the class of all approximations to the desired functions. So, given a set-like relation R and a class function $G : \mathbf{V} \to \mathbf{V}$ we defined $\mathfrak{F}$ to be the class of all functions g such that

- $\mathrm{dom}(g) = \mathrm{trcl}_R(a)$ for some a
- $g(x) = G(g \restriction \mathrm{pred}_R(x))$ whenever $x \in \mathrm{dom}(g)$

Give a detailed explanation why $\mathfrak{F}$ is a class. More precisely, explain how $\mathfrak{F}$ can be define by a formula in the LST.

2. Prove that the class of all transitive sets is a proper class.

Hint. Show that $\mathbf{V}$ can be constructed from transitive sets. Make this precise. You will need the proposition on transitive closures here.

3. Recall that a finite sequence is a function f with $\mathrm{dom}(f) \in \omega$. We often write $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ to denote the finite function f with domain n such that $f(i) = x_i$.

If A is a class, let A' be the class of all x such that there is a finite sequence $\langle x_0, x_1, \ldots, x_{n-1} \rangle$ satisfying:

- $x_0 \in A$
- $x_{n-1} = x$
- $x_{i+1} \mathrel{R} x_i$ for all $i < n$.

Prove that A' is really a class and that it the R-transitive closure of A. Also, use Exercise 3 from Homework 2 to prove that if A is a set then so is A'. This gives you a different proof of Proposition 1.25.

Hint. Verify the definition in the straightforward way.

In the following exercises you can use facts about transitive sets from the lecture and the previous homework assignments.

4. Directly from the definition of ordinal show that $\in$ is trichotomic on $\mathbf{On}$. That is, show that if x, y are ordinals then either $x \in y$ or $x = y$ or $y \in x$.

Hint. Show that if $y \neq x \cap y$ then $x = x \cap y \in y$. In order to see this, let a be the least element of $y - (x \cap y)$. Show that $a = x \cap y$. This takes some thought.

5. Prove the following facts about ordinals.

(a) If x is an ordinal then $S(x)$ is an ordinal.

(b) If a is a set of ordinals then $\bigcup a$ is an ordinal.

(c) If A is a proper class of ordinals then $\bigcup A = \mathbf{On}$.

Hint. Again, in (a) and (b) verify the definition of ordinal in the straightforward way. In (c), the point is to focus on $\supseteq$.