HOMEWORK 4

1. Prove the following.

(a) If α and β are ordinals and $\alpha \subseteq \beta$ then $\alpha \leq \beta$. So $\subseteq$ is the set-theoretical version of $\leq$ in the case of ordinals.

(b) If α is an ordinal then $S(\alpha)$ is the smallest ordinal larger than α.

(c) If A is a set of ordinals then $\bigcup A$ is the smallest ordinal larger than all elements of A. This justifies writing $\sup(A)$ instead.

Hint. Use (a) for the proof of (b) and (c).

2. Let R be a set-like relation.

(a) Show that the relation R is well-founded if and only if there is a function $f : \mathbf{V} \to \mathbf{On}$ such that for every $x, y \in \mathbf{V}$ we have

$$xRy \implies f(x) < f(y).$$

(b) Assume f is a function as in (a). Show that for every $x \in \mathbf{V}$ we have $\mathrm{rank}_R(x) \leq f(x)$.

(c) Assume the full ZF. Let

$$f(x) = \text{the least ordinal } \alpha \text{ such that } x \subseteq V_\alpha.$$

Show that $f(x) = \mathrm{rank}(x)$.

Hint. (a) is just straightforward application of the definition of well-foundedness and rank functions. (b) is proved by induction on R. To see (c) use the uniqueness part of the theorem on construction by recursion.

3. Work in ZF without the foundation axiom.

(a) Give a rigorous proof that $\mathbf{WF}$ is well-founded, i.e. that if $x \in \mathbf{WF}$ is nonempty then x has an $\in$-minimal element.

(b) Show that $V_\alpha \notin V_\alpha$ for all $\alpha \in \mathbf{On}$.

Hint. For (a) use the properties of the V_α hierarchy from the lecture.

4. Show that the lexicographic ordering induced by two well-orderings is a well-ordering.

Hint. This is a straightforward verification of the definition.

5. Prove the following facts about ordinal arithmetic. Let α and β be ordinals.

(a) $\alpha + \beta = \operatorname{otp}\Big((\{0\} \times \alpha) \cup (\{1\} \times \beta), <_{\mathrm{Lex}}\Big)$.

(b) $\alpha \cdot \beta = \operatorname{otp}(\beta \times \alpha, <_{\mathrm{Lex}})$

Hint. In either case proceed by recursion on β.