HOMEWORK 4

1. Prove the following.

(a) If $\alpha$ and $\beta$ are ordinals and $\alpha \subseteq \beta$ then $\alpha \leq \beta$. So $\subseteq$ is the set-theoretical version of $\leq$ in the case of ordinals.

(b) If $\alpha$ is an ordinal then $S(\alpha)$ is the smallest ordinal larger than $\alpha$.

(c) If $A$ is a set of ordinals then $\bigcup A$ is the smallest ordinal larger than all elements of $A$. This justifies writing $\text{sup}(A)$ instead.

**Hint.** Use (a) for the proof of (b) and (c).

2. Let $R$ be a set-like relation.

(a) Show that the relation $R$ is well-founded if and only if there is a function $f : V \rightarrow \text{On}$ such that for every $x, y \in V$ we have

$$xRy \implies f(x) < f(y).$$

(b) Assume $f$ is a function as in (a). Show that for every $x \in V$ we have $\text{rank}_R(x) \leq f(x)$.

(c) Assume the full ZF. Let

$$f(x) = \text{the least ordinal } \alpha \text{ such that } x \subseteq V_\alpha.$$ 

Show that $f(x) = \text{rank}(x)$.

**Hint.** (a) is just straightforward application of the definition of well-foundedness and rank functions. (b) is proved by induction on $R$. To see (c) use the uniqueness part of the theorem on construction by recursion.

3. Work in ZF without the foundation axiom.

(a) Give a rigorous proof that $\text{WF}$ is well-founded, i.e. that if $x \in \text{WF}$ is nonempty then $x$ has an $\in$-minimal element.

(b) Show that $V_\alpha \notin V_\alpha$ for all $\alpha \in \text{On}$. 

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**Hint.** For (a) use the properties of the $V_\alpha$ hierarchy from the lecture.

4. Show that the lexicographic ordering induced by two well-orderings is a well-ordering.

**Hint.** This is a straightforward verification of the definition.

5. Prove the following facts about ordinal arithmetic. Let $\alpha$ and $\beta$ be ordinals.

   (a) $\alpha + \beta = \text{otp}\left(\{0\} \times \alpha \cup \{1\} \times \beta, <_{\text{Lex}}\right)$.

   (b) $\alpha \cdot \beta = \text{otp}(\beta \times \alpha, <_{\text{Lex}})$

**Hint.** In either case proceed by recursion on $\beta$. 