HOMEWORK 5

1. Work in the full ZF with Axiom of Foundation. Prove that the following are equivalent.

(a) A is a transitive set all of whose elements are transitive.

(b) A is an ordinal.

Hint. Concerning the nontrivial part (a)$\Longrightarrow$(b): Assuming A is not an ordinal, look at some $\in$-minimal element of A that is not an ordinal.

2. Let A be a set and $\prec$ be a strict linear ordering on A. Show that the maximo-lexicographical ordering $\prec_{\mathrm{mlex}}$ induced on $A \times A$ is a linear ordering. Moreover, if $\prec$ is a well-ordering on A then $\prec_{\mathrm{mlex}}$ is a well-ordering on $A \times A$.

Hint. This is a straightforward verification of the definitions.

3. Define the following ordering $<^*$ on ${}^{<\omega}[\mathbf{On}]$.

$$a <^* b \iff (\exists \alpha \in b - a)\Big(b - (\alpha + 1) = a - (\alpha + 1)\Big).$$

(Draw the picture!) Show:

(a) $<^*$ is a set-like well-ordering on ${}^{<\omega}[\mathbf{On}]$.

(b) If κ is an infinite cardinal then the initial segment $\langle \kappa \times \kappa, <^* \cap (\kappa \times \kappa) \rangle$ has order type κ.

Hint. This is a variation on the proof of Problem 2 above.

4. Let α and β be infinite limit ordinals and $h : \alpha \to \beta$ be a strictly increasing cofinal map. Show that $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

Hint. To see that $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$ use the theorem from the lecture: If there is a cofinal map $g : \gamma \to \alpha$ then there is a strictly increasing cofinal map $g' : \gamma' \to \alpha$ where $\gamma' \leq \gamma$.