HOMEWORK 1

1. Work in ZF. Let κ be an infinite cardinal. Prove that the following two facts are equivalent.

(a) $\mathcal{P}(\alpha)$ can be well-ordered for every ordinal α.

(b) Axiom of Choice, or equivalently, every set can be well-ordered.

Hint. By induction on α prove that each V_α can be well-ordered. (Why is this sufficient?) The successor step of the induction is easy. To see that V_α is well-orderable for limit α use the fact that each $x \in V_\alpha$ can be encoded into a subset of $\beta \times \beta$ for some ordinal β via an obvious bijection. This coding makes use of the Mostowski collapsing theorem. The argument is similar to the argument for HW3, Problem 3(f) from winter assignment.

2. Let $\mathcal{L}$ be a first-order language and Γ be a set of $\mathcal{L}$-sentences. Define the following binary relation on the set of all $\mathcal{L}$-sentences:

$$\varphi \sim \psi \iff \Gamma \vdash \varphi \leftrightarrow \psi.$$

Show that $\sim$ is an equivalence relation on the set of all $\mathcal{L}$-sentences. Let B be the corresponding quotient set.

(a) Assume that Γ is consistent. On B define operations $\wedge$, $\vee$, $\mathbf{0}$, $\mathbf{1}$ and $'$ as follows: $[\varphi] \wedge [\psi] = [\varphi \& \psi]$, $[\varphi] \vee [\psi] = [\varphi \vee \psi]$, $\mathbf{0} = [\sigma \& \neg\sigma]$, $\mathbf{1} = [\sigma \vee \neg\sigma]$ and $[\varphi]' = [\neg\varphi]$. Show that the structure $\mathbb{B}_\Gamma$ with support B and the above operations is a Boolean algebra. This algebra is called the **Lindenbaum algebra** of Γ. Find the definition of the corresponding partial ordering in terms of provability from Γ.

(b) For any $\mathcal{L}$-sentence σ prove that $\Gamma \vdash \sigma$ iff $[\sigma] = \mathbf{1}$, and Γ is consistent with σ iff $[\sigma] \neq \mathbf{0}$.

(c) More generally, let Σ be a set of $\mathcal{L}$-sentences. Show that

$$S_\Sigma = \{[\sigma] \mid \sigma \in \Sigma\}$$

induces a filter F_Σ on B_Γ in a natural manner if and only if $\Gamma \cup \Sigma$ is consistent. Find out how is this filter induced and write down a formal definition of F_Σ from S_Σ.

(d) Let Σ be a set of $\mathcal{L}$-sentences such that $\Gamma \cup \Sigma$ is consistent. Show that the Lindenbaum algebra $B_{\Gamma \cup \Sigma}$ is isomorphic to the qotient $\mathbb{B}_\Gamma / F_\Sigma$.

(e) Let $\mathbb{B}$ be the Lindenbaum algebra for the axioms of predicate logic in $\mathcal{L}$, that is, $\mathbb{B} = \mathbb{B}_\Gamma$ where Γ is just the set of all axioms of predicate logic. Let Σ be a consistent set of $\mathcal{L}$-sentences. Show that Σ is complete iff F_Σ is an ultrafilter on $\mathbb{B}$.

(f) Determine the Lindenbaum algebra of a complete set of sentences.