**HOMEWORK 1**

1. Work in ZF. Let $\kappa$ be an infinite cardinal. Prove that the following two facts are equivalent.

   (a) $\mathcal{P}(\alpha)$ can be well-ordered for every ordinal $\alpha$.

   (b) Axiom of Choice, or equivalently, every set can be well-ordered.

   **Hint.** By induction on $\alpha$ prove that each $V_\alpha$ can be well-ordered. (Why is this sufficient?) The successor step of the induction is easy. To see that $V_\alpha$ is well-orderable for limit $\alpha$ use the fact that each $x \in V_\alpha$ can be encoded into a subset of $\beta \times \beta$ for some ordinal $\beta$ via an obvious bijection. This coding makes use of the Mostowski collapsing theorem. The argument is similar to the argument for HW3, Problem 3(f) from winter assignment.

2. Let $\mathcal{L}$ be a first-order language and $\Gamma$ be a set of $\mathcal{L}$-sentences. Define the following binary relation on the set of all $\mathcal{L}$-sentences:

   $$\varphi \sim \psi \iff \Gamma \vdash \varphi \leftrightarrow \psi.$$ 

   Show that $\sim$ is an equivalence relation on the set of all $\mathcal{L}$-sentences. Let $B$ be the corresponding quotient set.

   (a) Assume that $\Gamma$ is consistent. On $B$ define operations $\land$, $\lor$, $0$, $1$ and $'$ as follows: $[\varphi \land [\psi] = [\varphi \land \psi]$, $[\varphi \lor [\psi] = [\varphi \lor \psi]$, $0 = [\varphi \land \neg \varphi]$, $1 = [\varphi \lor \neg \varphi]$ and $[\varphi'] = [\neg \varphi]$. Show that the structure $B_\Gamma$ with support $B$ and the above operations is a Boolean algebra. This algebra is called the **Lindenbaum algebra** of $\Gamma$. Find the definition of the corresponding partial ordering in terms of provability from $\Gamma$.

   (b) For any $\mathcal{L}$-sentence $\sigma$ prove that $\Gamma \vdash \sigma$ iff $[\sigma] = 1$, and $\Gamma$ is consistent with $\sigma$ iff $[\sigma] \neq 0$.

   (c) More generally, let $\Sigma$ be a set of $\mathcal{L}$-sentences. Show that

   $$S_\Sigma = \{[\sigma] \mid \sigma \in \Sigma\}$$

   induces a filter $F_\Sigma$ on $B_\Gamma$ in a natural manner if and only if $\Gamma \cup \Sigma$ is consistent. Find out how is this filter induced and write down a formal definition of $F_\Sigma$ from $S_\Sigma$. 

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(d) Let $\Sigma$ be a set of $\mathcal{L}$-sentences such that $\Gamma \cup \Sigma$ is consistent. Show that the Lindenbaum algebra $B_{\Gamma \cup \Sigma}$ is isomorphic to the quotient $B_{\Gamma} / F_{\Sigma}$.

(e) Let $\mathcal{B}$ be the Lindenbaum algebra for the axioms of predicate logic in $\mathcal{L}$, that is, $\mathcal{B} = B_{\Gamma}$ where $\Gamma$ is just the set of all axioms of predicate logic. Let $\Sigma$ be a consistent set of $\mathcal{L}$-sentences. Show that $\Sigma$ is complete iff $F_{\Sigma}$ is an ultrafilter on $\mathcal{B}$.

(f) Determine the Lindenbaum algebra of a complete set of sentences.