HOMEWORK 2

1. Let $\mathcal{L}$ be a language.

(a) For any $n \in \omega - \{0\}$ find a sentence σ_n such that for any $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma_n$ if and only if

  (i) $\mathfrak{A}$ has at most n elements;

  (ii) $\mathfrak{A}$ has at least n elements;

  (iii) $\mathfrak{A}$ has precisely n elements.

(b) Show that there is no $\mathcal{L}$-sentence σ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is finite.

(c) Is there an $\mathcal{L}$-sentence σ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is infinite? Justify your answer.

(d) Find a set of $\mathcal{L}$-sentences Σ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \Sigma$ iff $\mathfrak{A}$ is infinite.

(e) Is there a set of $\mathcal{L}$-sentences Σ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \Sigma$ iff $\mathfrak{A}$ is finite? Justify your answer.

Hint. (b) and (e) rely on the Compactness Theorem.

A class of structures $\mathbf{K}$ is **elementary** just in case that there is a set of $\mathcal{L}$-sentences Σ such that

$$\mathbf{K} = \{\mathfrak{A} \mid \mathfrak{A} \models \Sigma\};$$

in this case we say that Σ **axiomatizes** $\mathbf{K}$, and write $\mathbf{K}_\Sigma$ for $\mathbf{K}$.

2. In either of the following cases find a set of $\mathcal{L}$-sentences, if possible a finite one, that axiomatizes the respective class of structures.

(a) $\mathcal{L}$ has only one binary predicate symbol $\dot{<}$ and $\mathbf{K}$ is the class of all strict linear orderings.

(b) $\mathcal{L}$ has only one binary predicate symbol $\dot{<}$ and $\mathbf{K}$ is the class of all strict dense linear orderings without endpoints.

(c) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations.

(d) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations all of whose equivalence classes have precisely two elements.

(e) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations that have infinitely many equivalence classes and all equivalence classes have infinitely many elements.

3. In the following decide whether the given class of $\mathcal{L}$-structures is elementary.

(a) $\mathcal{L}$ has only one binary predicate symbol $\dot{<}$ and $\mathbf{K}$ is the class of all well-orderings.

(b) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations which have only finite equivalence classes.

(c) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations which have infinitely many equivalence classes.

(d) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations all of whose equivalence classes are infinite.

(e) $\mathcal{L}$ has only one binary predicate symbol $\dot{E}$ and $\mathbf{K}$ is the class of all equivalence relations that have infinitely many infinite equivalence classes (but not necessarily all all equivalence classes are infinite).

(f) $\mathcal{L}$ has only one constant symbol $\dot{0}$ and one binary predicate symbol $\dot{+}$ and $\mathbf{K}$ is the class of all torsion-free abelian groups.

(g) $\mathcal{L}$ has only one constant symbol $\dot{0}$ and one binary predicate symbol $\dot{+}$ and $\mathbf{K}$ is the class of all abelian groups with torsion.

Hint. Some of the negative answers rely on the compactness theorem, some not. Try to find out.