HOMEWORK 2

1. Let $\mathcal{L}$ be a language.

   (a) For any $n \in \omega - \{0\}$ find a sentence $\sigma_n$ such that for any $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma_n$ if and only if

      (i) $\mathfrak{A}$ has at most $n$ elements;

      (ii) $\mathfrak{A}$ has at least $n$ elements;

      (iii) $\mathfrak{A}$ has precisely $n$ elements.

   (b) Show that there is no $\mathcal{L}$-sentence $\sigma$ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is finite.

   (c) Is there an $\mathcal{L}$-sentence $\sigma$ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is infinite? Justify your answer.

   (d) Find a set of $\mathcal{L}$-sentences $\Sigma$ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \Sigma$ iff $\mathfrak{A}$ is infinite.

   (e) Is there a set of $\mathcal{L}$-sentences $\Sigma$ such that for every $\mathcal{L}$-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \Sigma$ iff $\mathfrak{A}$ is finite? Justify your answer.

   **Hint.** (b) and (e) rely on the Compactness Theorem.

   A class of structures $\mathbf{K}$ is **elementary** just in case that there is a set of $\mathcal{L}$-sentences $\Sigma$ such that

   $$\mathbf{K} = \{ \mathfrak{A} | \mathfrak{A} \models \Sigma \};$$

   in this case we say that $\Sigma$ **axiomatizes** $\mathbf{K}$, and write $\mathbf{K}_\Sigma$ for $\mathbf{K}$.

2. In either of the following cases find a set of $\mathcal{L}$-sentences, if possible a finite one, that axiomatizes the respective class of structures.

   (a) $\mathcal{L}$ has only one binary predicate symbol $\prec$ and $\mathbf{K}$ is the class of all strict linear orderings.

   (b) $\mathcal{L}$ has only one binary predicate symbol $\prec$ and $\mathbf{K}$ is the class of all strict dense linear orderings without endpoints.
(c) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations.

(d) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations all of whose equivalence classes have precisely two elements.

(e) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations that have infinitely many equivalence classes and all equivalence classes have infinitely many elements.

3. In the following decide whether the given class of \( \mathcal{L} \)-structures is elementary.

(a) \( \mathcal{L} \) has only one binary predicate symbol \( \prec \) and \( \mathbf{K} \) is the class of all well-orderings.

(b) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations which have only finite equivalence classes.

(c) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations which have infinitely many equivalence classes.

(d) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations all of whose equivalence classes are infinite.

(e) \( \mathcal{L} \) has only one binary predicate symbol \( \tilde{E} \) and \( \mathbf{K} \) is the class of all equivalence relations that have infinitely many infinite equivalence classes (but not necessarily all all equivalence classes are infinite).

(f) \( \mathcal{L} \) has only one constant symbol \( \hat{0} \) and one binary predicate symbol \( \hat{+} \) and \( \mathbf{K} \) is the class of all torsion-free abelian groups.

(g) \( \mathcal{L} \) has only one constant symbol \( \hat{0} \) and one binary predicate symbol \( \hat{+} \) and \( \mathbf{K} \) is the class of all abelian groups with torsion.

\textbf{Hint.} Some of the negative answers rely on the compactness theorem, some not. Try to find out.