HOMEWORK 3

1. Let $\mathcal{L}$ be a first-order language. It is possible to imitate the proof of the Completeness Theorem to obtain a direct proof of the Compactness Theorem. This way we do not obtain the Completeness theorem, but we obtain a proof of the Compactness theorem that avoids any reference to syntax, i.e. to axioms, rules and proofs. Try to carry out the proof. Below are the necessary requisities.

We say that a set of $\mathcal{L}$-sentences Γ is **satisfiable** or **has a model** just in case that there is some $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$; such a structure $\mathfrak{A}$ is called a **model** of Γ.

Compactness Theorem. Let Γ be a set of $\mathcal{L}$-sentences. If every finite $\Delta \subseteq \Gamma$ has a model then also Γ has a model.

Definition 1. We say that a set of $\mathcal{L}$-sentences Γ:

- Is **complete** just in case that for every $\mathcal{L}$-sentence σ we have $\sigma \in \Gamma$ or $\neg\sigma \in \Gamma$.
- Has Henkin constants just in case that for every formula $\varphi(x)$ with a single free variable x we have: If $(\exists x)\varphi(x) \in \Gamma$ then there is some constant symbol c in the language $\mathcal{L}$ such that $\varphi(c) \in \Gamma$.

Prove the following analogue to Theorem 7.8 from the lecture:

Theorem 7.8.A Assume Γ is a set of $\mathcal{L}$-sentences satisfying the following conditions.

(a) Every finite $\Delta \subseteq \Gamma$ is satisfiable.

(b) Γ is complete.

(c) Γ has Henkin constants.

Then Γ has a model whose cardinality is at most that of $C^{\mathcal{L}}$ where $C^{\mathcal{L}}$ is the set of all constant symbols of $\mathcal{L}$.

Here completeness and Henkin constants are understood in the sense of Definition 1.

Hint. Define a binary relation $\sim$ on the set $C^{\mathcal{L}}$ as follows.

$$c \sim c' \iff \Gamma \models c \doteq c'.$$

From now on imitate the proof of Theorem 7.8 from the lecture, replacing arguments for $\vdash$ in that proof by arguments for $\models$; this requires modifying the arguments appropriately.

Next imitate the proof of Theorem 7.10 from the lecture, again by replacing arguments for $\vdash$ by arguments for $\models$ in the proof of Theorem 7.10 and amending the arguments appropriately. You obtain:

Theorem 7.10.A. Let Γ be a set of $\mathcal{L}$ sentences such that every finite $\Delta \subseteq \Gamma$ is satisfiable. Then there is a language $\mathcal{L}^*$ and a set of $\mathcal{L}^*$-sentences Γ^* such that

(a) $|\mathcal{L}^*| \leq \max\{|\mathcal{L}|, \aleph_0\}$.

(b) $\Gamma \subseteq \Gamma^*$.

(c) Every finite $\Delta \subseteq \Gamma^*$ is satisfiable.

(d) Γ^* is complete.

(e) Γ^* has Henkin constants.

Again, completeness and Henkin constants are understood in the sense of Definition 1.