HOMEWORK 4

1. Modify our logical calculus by deleting the quantifier rule and adding the following axiom schema:

$$(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$$

where x has no free occurence in ψ. Do we get a useful deductive calculus this way? Make this statement precise and find and justify the answer.

The following exercises are intended to practise arguments that involve applications of the Compactness Theorem.

2. Let Σ be a set of $\mathcal{L}$-sentences and σ be an $\mathcal{L}$-sentence. Prove: If $\Sigma \models \sigma$ then there is a finite $\Delta \subseteq \Sigma$ such that $\Delta \models \Sigma$.

3. Let Γ and Γ' be two sets of $\mathcal{L}$-sentences.

(a) Assume that $\Gamma \cup \Gamma'$ has no model. Show that there is an $\mathcal{L}$-sentence σ such that $\Gamma \models \sigma$ and $\Gamma' \models \neg\sigma$.

(b) Assume for any $\mathcal{L}$-structure $\mathfrak{A}$ we have

$$\mathfrak{A} \models \Gamma \quad \text{iff} \quad \mathfrak{A} \not\models \Gamma'$$

or equivalently

$$\mathbf{K}_\Gamma \cap \mathbf{K}_{\Gamma'} = \varnothing \quad \text{and} \quad \mathbf{K}_\Gamma \cup \mathbf{K}_{\Gamma'} = \text{ the class of all } \mathcal{L}\text{-structures.}$$

Show that both Γ and Γ' are finitely axiomatizable.

4. Let $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \Gamma_n \subseteq \cdots$ be an increasing sequence of sets of $\mathcal{L}$-sentences satisfying the following requirement for each $n \in \omega$:

- There is some $\sigma \in \Gamma_{n+1}$ such that $\Gamma_n \not\models \sigma$.

Intuitively, this requirement expresses that Γ_n relly increase in terms of strentgh. Let

$$\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n.$$

Show that Γ is consistent and it is not finitely axiomatizable.

5. Let σ be an $\mathcal{L}$-sentence. The **finite spectrum** of σ, denoted by $\mathsf{FSP}(\sigma)$ is the set of all positive integers n such that there is a finite model of σ of size n. Symbolically,

$$\mathsf{FSP}(\sigma) = \{n \in \mathbb{N} \,|\, \mathfrak{A} \models \sigma \text{ for some } \mathcal{L}\text{-structure } \mathfrak{A} \text{ of size } n\}.$$

In the following we write $\mathbb{N}$ for ω and $\mathbb{N}^+$ for $\omega - \{0\}$.

(a) Let $\mathcal{L}$ contain only one binary predicate symbol $\dot{E}$. Let σ be the sentence in $\mathcal{L}$ expressing that $\dot{E}$ is an equivalence relation all of whose equivalence classes have size 2. Show that $\mathsf{FSP}(\sigma)$ is the set of all even positive integers.

In each of the following cases find a finite language $\mathcal{L}$ and an $\mathcal{L}$-sentence σ such that the given set of integers is the finite spectrum of σ.

(b) $\{n \in \mathbb{N} \,|\; n \text{ is odd}\}$

(c) $\{2^n \,|\, n \in \mathbb{N}^+\}$

(d) $\{2^n \cdot 3^m \,|\, m, n \in \mathbb{N}^+\}$

(e) $\{p \in \mathbb{N} \,|\; p \text{ is a prime}\}$

(f) $\{c \in \mathbb{N}^+ \;|\; c \text{ is a composite number}\}$

Hint. In (c) – (f) look at the right algebraic structures. A composite number is a number that can be expressed as the product $a \cdot b$ where $a, b \in \mathbb{N}^+$.