HOMEWORK 5

1. Let $\mathcal{L}$ be a finite language. Prove that for any two **finite** $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ the following is true:

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{iff} \quad \mathfrak{A}, \mathfrak{B} \text{ are isomorphic.}$$

Recall that $\mathfrak{A} \equiv \mathfrak{B}$ means that $\mathfrak{A}, \mathfrak{B}$ are elementarily equivalent.

2. Recall that if Σ is a consistent set of $\mathcal{L}$-sentences we say that $\bar{\Sigma}$ is an axiomatization of Σ just in case that for all $\mathcal{L}$-sentences σ we have

$$\Sigma \models \sigma \iff \bar{\Sigma} \models \sigma.$$

A consistent set Σ of $\mathcal{L}$-sentences is finitely axiomatizable iff there is some finite axiomatization of Σ.

Assume Σ is a consistent finitely axiomatizable set of $\mathcal{L}$-sentences and that Σ does not have any finitely axiomatizable **complete** extension. Show that there are $2^{\aleph_0}$ complete extensions of Σ.

Hint. Notice that Σ is not complete, so there is an $\mathcal{L}$-sentence σ_0 such that $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$ are both consistent. Iterate this to obtain $\mathcal{L}$-sentences σ_n for all $n \in \omega$. These sentences give rise to a tree of height ω in the natural way. Look at the infinite branches of this tree. Use the compactness theorem. Formulate everything rigorously.

3. Let $\mathcal{L}$ be a countable language and Σ be a complete set of $\mathcal{L}$-sentences. Show that there is a model $\mathfrak{A} \models \Sigma$ of cardinality $\leq 2^{\aleph_0}$ with the following property.

If $\mathfrak{B} \models \Sigma$ and $B' \subseteq B$ is countable then there is some $A' \subseteq A$ such that $\mathfrak{B}_B$ is elementarily equivalent to $\mathfrak{A}_A$.

Hint. Show that there are at most $2^{\aleph_0}$ countable models of Σ modulo isomorphism. Then for each such model $\mathfrak{C}$ and each $C' \subseteq C$ consider the diagram $\mathrm{Th}(\mathfrak{C}_C)$. Let Γ be the union of all such diagrams. Show that Γ is consistent, here use an argument similar to that in the proof of Claim 3 in the proof Joint Consistency Theorem from the lecture. Then show that if $\mathfrak{A} \models \Gamma$ then $\mathfrak{A}$ is as required. To translate from uncountable structures to countable structures use the downward Löwenheim-Skolem theorem.

4. The language $\mathcal{L}$ for the theory of linear orderings has only one binary predicate symbol $\dot{<}$.

(a) Write down a finite list Σ of $\mathcal{L}$-sentences that expresses the property of being a dense linear ordering without endpoints. That is, for every $\mathcal{L}$-structure $\mathfrak{A}$ we have:

$$\mathfrak{A} \models \Sigma \quad \text{iff} \quad \mathfrak{A} \text{ is a dense linear ordering without endpoints.}$$

Here "dense" means that between any two distinct elements there is a third one. An endpoint is either the largest element or the smallest element.

(b) Show that Σ is complete.

Hint. Regarding (b), show that if $\mathfrak{A} \models \Sigma$ then $\mathfrak{A} \equiv (\mathbb{Q}, <)$ where $\mathbb{Q}$ is the set of all rational numbers and $<$ is the usual ordering on rational numbers. To do this, use the downward Löwenheim-Skolem Theorem to show that $\mathfrak{A}$ has a countable elementary substructure $\mathfrak{A}'$ and then show that $\mathfrak{A}'$ is isomorphic to $(\mathbb{Q}, <)$. The isomorphism is constructed inductively in a back-and-forth manner, this is a construction due to Cantor that you should know from analysis.

5. Let $\mathcal{L}$ be a language with one binary function symbol $\dot{+}$ and one constant symbol $\dot{0}$. Any Abelian group can be viewed as an $\mathcal{L}$-structure where $\dot{+}$ is interpreted as addition and $\dot{0}$ as the identity element of the group.

Let $\mathbb{Z}$ be the Abelian group of all integers. Show that $\mathbb{Z}$ is not elementarily equivalent with $\mathbb{Z} \oplus \mathbb{Z}$.