HOMEWORK 5

1. Let $\mathcal{L}$ be a finite language. Prove that for any two finite $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ the following is true:

$$\mathfrak{A} \equiv \mathfrak{B} \iff \mathfrak{A}, \mathfrak{B} \text{ are isomorphic.}$$

Recall that $\mathfrak{A} \equiv \mathfrak{B}$ means that $\mathfrak{A}, \mathfrak{B}$ are elementarily equivalent.

2. Recall that if $\Sigma$ is a consistent set of $\mathcal{L}$-sentences we say that $\bar{\Sigma}$ is an axiomatization of $\Sigma$ just in case that for all $\mathcal{L}$-sentences $\sigma$ we have

$$\Sigma \models \sigma \iff \bar{\Sigma} \models \sigma.$$  

A consistent set $\Sigma$ of $\mathcal{L}$-sentences is finitely axiomatizable iff there is some finite axiomatization of $\Sigma$.

Assume $\Sigma$ is a consistent finitely axiomatizable set of $\mathcal{L}$-sentences and that $\Sigma$ does not have any finitely axiomatizable complete extension. Show that there are $2^{\aleph_0}$ complete extensions of $\Sigma$.

**Hint.** Notice that $\Sigma$ is not complete, so there is an $\mathcal{L}$-sentence $\sigma_0$ such that $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg \sigma\}$ are both consistent. Iterate this to obtain $\mathcal{L}$-sentences $\sigma_n$ for all $n \in \omega$. These sentences give rise to a tree of height $\omega$ in the natural way. Look at the infinite branches of this tree. Use the compactness theorem. Formulate everything rigorously.

3. Let $\mathcal{L}$ be a countable language and $\Sigma$ be a complete set of $\mathcal{L}$-sentences. Show that there is a model $\mathfrak{A} \models \Sigma$ of cardinality $\leq 2^{\aleph_0}$ with the following property.

If $\mathfrak{B} \models \Sigma$ and $B' \subseteq B$ is countable then there is some $A' \subseteq A$ such that $\mathfrak{B}_{B}$ is elementarily equivalent to $\mathfrak{A}_{A}$.

**Hint.** Show that there are at most $2^{\aleph_0}$ countable models of $\Sigma$ modulo isomorphism. Then for each such model $\mathfrak{C}$ and each $C' \subseteq C$ consider the diagram $Th(\mathfrak{C}_{C'})$. Let $\Gamma$ be the union of all such diagrams. Show that $\Gamma$ is consistent, here use an argument similar to that in the proof of Claim 3 in the proof Joint Consistency Theorem from the lecture. Then show that if $\mathfrak{A} \models \Gamma$ then $\mathfrak{A}$ is as required. To translate from uncountable structures to countable structures use the downward Löwenheim-Skolem theorem.
4. The language $L$ for the theory of linear orderings has only one binary predicate symbol $\prec$.

   (a) Write down a finite list $\Sigma$ of $L$-sentences that expresses the property of being a dense linear ordering without endpoints. That is, for every $L$-structure $\mathfrak{A}$ we have:

   $$\mathfrak{A} \models \Sigma \quad \text{iff} \quad \mathfrak{A} \text{ is a dense linear ordering without endpoints.}$$

   Here “dense” means that between any two distinct elements there is a third one. An endpoint is either the largest element or the smallest element.

   (b) Show that $\Sigma$ is complete.

   **Hint.** Regarding (b), show that if $\mathfrak{A} \models \Sigma$ then $\mathfrak{A} \cong (\mathbb{Q}, \lt)$ where $\mathbb{Q}$ is the set of all rational numbers and $\lt$ is the usual ordering on rational numbers. To do this, use the downward Löwenheim-Skolem Theorem to show that $\mathfrak{A}$ has a countable elementary substructure $\mathfrak{A}'$ and then show that $\mathfrak{A}'$ is isomorphic to $(\mathbb{Q}, \lt)$. The isomorphism is constructed inductively in a back-and-forth manner, this is a construction due to Cantor that you should know from analysis.

5. Let $L$ be a language with one binary function symbol $+$ and one constant symbol $0$. Any Abelian group can be viewed as an $L$-structure where $+$ is interpreted as addition and $0$ as the identity element of the group.

   Let $\mathbb{Z}$ be the Abelian group of all integers. Show that $\mathbb{Z}$ is not elementarily equivalent with $\mathbb{Z} \oplus \mathbb{Z}$. 