HOMEWORK 6

Given an $\mathcal{L}$-structure $\mathfrak{A}$, an **automorphism** of $\mathfrak{A}$ is any isomorphism $h : \mathfrak{A} \to \mathfrak{A}$. Given any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_\ell)$ and any $a_1, \ldots, a_\ell \in A$ we then have

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_\ell] \quad \text{iff} \quad \mathfrak{A} \models \varphi[h(a_1), \ldots, h(a_\ell)]$$

and hence if $R \subset {}^\ell A$ is an ℓ-place relation that is lightface definable in $\mathfrak{A}$ we get

$$\langle a_1, \ldots, a_\ell \rangle \in R \quad \text{iff} \quad \langle h(a_1), \ldots, h(a_\ell) \rangle \in R$$

This fact can be used to prove that certain relations are not definable in given structures.

1. In the following decide the definability of given relations in given structures. For affirmative answers give the definitions of the respective relations. For negative answers use the above fact on automorphisms; in each case construct an automorphism that witnesses that the given relation is not definable.

(a) $\mathcal{L}$ is the language $\{\dot{<}\}$ for linear orderings. Show that $\varnothing$ and $\mathbb{Q}$ are the only subsets definable in the structure $(\mathbb{Q}, <)$.

(b) $\mathcal{L}$ is the language $\{\dot{<}\}$ for linear orderings and $\mathcal{L}_\mathbb{Q}$ is its usual extension containing constant symbols denoting elements of $\mathbb{Q}$. Show that a set $A \subseteq \mathbb{Q}$ is definable in the structure $(\mathbb{Q}, <)_\mathbb{Q}$ if and only if A is a finite union of intervals (possibly unbounded ones). Here we also allow degenerate intervals, so the closed interval $[q, q]$ is considered to be equal to the singleton $\{q\}$.

Hint. To see that finite unions are the only sets definable in the structure $(\mathbb{Q}, <)_\mathbb{Q}$, first show the following. Given a set $K \subseteq \mathbb{Q}$, the language $\mathcal{L}_{\mathbb{Q},K}$ is the usual extension of $\mathcal{L}$ that contains constants denoting elements of K. Show that if K is finite then a set $A \subseteq \mathbb{Q}$ is definable in $(\mathbb{Q}, <)_{\mathbb{Q},K}$ if and only if A is a finite union of intervals whose endpoints are elements of K. (Here $(\mathbb{Q}, <)_{\mathbb{Q},K}$ is the obvious structure for the language $\mathcal{L}_{\mathbb{Q},K}$.)

(c) $\mathcal{L}$ contains only one binary function symbol $\dot{+}$. Let $\mathfrak{A}$ be an Abelian group viewed as a structure for $\mathcal{L}$. Show that 0 is definable in $\mathfrak{A}$. Also show that if $a \in \mathfrak{A}$ is such that $a + a \neq 0$ then a is not definable in $\mathfrak{A}$.

(d) Let $\mathcal{L}$ be as in (c). Let $\mathcal{L}'$ be a language with one binary function symbol $\dot{+}$, one unary function symbol $\dot{-}$ and one constant symbol $\dot{0}$. We can veiw Abelian groups as $\mathcal{L}'$-structures where the above symbols are interpreted as addition, inversion and 0. Show that if $\mathfrak{A}$ is an Abelian group and $X \subseteq A$ is definable in the sense of $\mathcal{L}'$ then X is also definable in the sense of $\mathcal{L}$.

(e) This is complementary to (c). Let $\mathcal{L}$ be as in (c). Notice that every element a of the group $\mathbb{Z}_2$ satisfies the equation $a + a = 0$. However, show that every element of $\mathbb{Z}_2$ is definable. On the other hand, every element a of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ also satisfies the equation $a + a = 0$. Show that the only definable element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is 0.

(f) Let $\mathcal{L}$ contain one binary function symbol $\dot{\cdot}$, which is interpreted in the set $\mathbb{N}$ of all natural numbers as multiplications. Show that addition is not definable in the $\mathcal{L}$-structure $(\mathbb{N}, \cdot)$, i.e. the relation

$$\{\langle x, y, z\rangle \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid x + y = z\}$$

is not definable in $(\mathbb{N}, \cdot)$.

Hint. Look at the prime decompositions of natural numbers.

2. Let Σ be a set of $\mathcal{L}$-sentences and $\varphi(x_1, \ldots, x_\ell)$ be an $\mathcal{L}$-formula. Let $c_1, \ldots, c_\ell$ be new constant symbols and $\mathcal{L}'$ be the language obtained by adding the symbols $c_1, \ldots, c_\ell$ to $\mathcal{L}$. Show that

$$\Sigma \models_{\mathcal{L}} (\forall x_1) \cdots (\forall x_\ell) \varphi(x_1, \ldots, x_\ell)$$

if and only if

$$\Sigma \models_{\mathcal{L}'} \varphi(c_1, \ldots, c_\ell).$$

This is just a direct consequence of the definition of $\models$.

3. Let $\mathcal{L}$ be a language and $\langle \mathfrak{A}_i \mid i \in I\rangle$ be a directed elementary system of $\mathcal{L}$-structures. So if $\lhd$ is the corresponding directed partial ordering on I then

$$i \lhd j \implies \mathfrak{A}_i \prec \mathfrak{A}_j.$$

Recall the definition of the union of the directed system of structures from the lecture. We let $\mathfrak{A}$ be the $\mathcal{L}$-structure whose support A is the union of all supports A_i of structures $\mathfrak{A}_i$ and interpret the $\mathcal{L}$-symbols as follows:

- If c is a constant symbol then $c^{\mathfrak{A}} = c^{\mathfrak{A}_i}$ for all $i \in I$.
- If F is an ℓ-place function symbol then

  $$F^{\mathfrak{A}}(a_1, \ldots, a_\ell) = a$$

  if and only if

  $$F^{\mathfrak{A}_i}(a_1, \ldots, a_\ell) = a \quad \text{for all } i \in I \text{ such that } a_1, \ldots, a_\ell, a \in A_i.$$

- If P is an ℓ-place predicate symbol then

  $$\langle a_1, \ldots, a_\ell \rangle \in P^{\mathfrak{A}}$$

  if and only if

  $$\langle a_1, \ldots, a_\ell \rangle \in P^{\mathfrak{A}_i} \quad \text{for all } i \in I \text{ such that } a_1, \ldots, a_\ell \in A_i.$$

Prove the theorem from the lecture on unions of directed systems:

(a) Prove that the interpretations of symbol have correct definitions; state what "correct" means in the case of each symbol and prove the corresponding statement concerning "correctness".

(b) Prove by induction on complexity of formulae that for each $i \in I$ we have $\mathfrak{A}_i \prec \mathfrak{A}$.

(c) Prove that if $\mathfrak{A}'$ is an $\mathcal{L}$-structure such that $\mathfrak{A}_i \prec \mathfrak{A}'$ for all $i \in I$ then $\mathfrak{A} \prec \mathfrak{A}'$.

4. Let $\mathcal{L}$ be a language consisting of one binary predicate symbol $\dot{<}$ and let $\mathfrak{N}$ be the $\mathcal{L}$-structure with support ω and with the interpretation $\dot{<}^{\mathfrak{N}}$ being the usual ordering on natural numbers. For any model $\mathfrak{A}$ of $\mathrm{Th}(\mathfrak{N})$ let $\sim^{\mathfrak{A}}$ be the binary relation on A defined by

$$a \sim^{\mathfrak{A}} b \iff \text{there are only finitely many } c \in A \text{ such that } a \dot{<}^{\mathfrak{A}} c \dot{<}^{\mathfrak{A}} b.$$

Then $\sim^{\mathfrak{A}}$ is an equivalence relation on A. On the quotient set $A/\sim^{\mathfrak{A}}$ define a binary relation $\ll^{\mathfrak{A}}$ as follows:

$$[a] \ll^{\mathfrak{A}} [b] \iff a \dot{<}^{\mathfrak{A}} b \text{ and } [a] \neq [b].$$

So $[a] \ll^{\mathfrak{A}} [b]$ just in case that $a \dot{<}^{\mathfrak{A}} b$ and there are infinitely many $c \in A$ such that $a \dot{<}^{\mathfrak{A}} c \dot{<}^{\mathfrak{A}} b$.

(a) Use elementary diagrams to show that for every $\mathcal{L}$-structure $\mathfrak{A}$ that is a model of $\mathrm{Th}(\mathfrak{N})$ there is an $\mathcal{L}$-structure $\mathfrak{A}'$ such that:

   (i) $\mathfrak{A} \prec \mathfrak{A}'$.

   (ii) There is some $c \in A'$ such that for every $a \in A$ we have $[a] \ll^{\mathfrak{A}'} [c]$.

   (iii) If $a, b \in A$ are such that $[a] \ll^{\mathfrak{A}} [b]$ then there is some $c \in A'$ such that $[a] \ll^{\mathfrak{A}'} [c] \ll^{\mathfrak{A}'} [b]$.

(b) Use elementary chains to show that there is an $\mathcal{L}$-structure $\mathfrak{A}^*$ such that $\mathfrak{A} \prec \mathfrak{A}^*$ and the linear ordering $\ll^{\mathfrak{A}^*}$ on $A^*/\sim^{\mathfrak{A}^*}$ is dense and does not have a largest element. (Here A^* is the support of $\mathfrak{A}^*$.) Recall that a linear ordering is dense iff between any two distinct elements there is a third one.

5. Show that the Robinson Joint Consistency Theorem follows from the Craig Interpolation Lemma.