HOMEWORK 7

1. Assume AC. Let $\mathcal{L}$ be a countable language and $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure. Show that there is a club $\mathcal{C}$ in $\mathcal{P}(\mathfrak{A})$ in the sense of Woodin’s definition such that every $\bar{A} \in \mathcal{C}$ induces an elementary substructure of $\mathfrak{A}$.

   **Hint.** Consider Skolem functions for $\mathfrak{A}$.

   For all problems concerning ultrapowers assume that the structures in question have Skolem functions.

2. Recall that an ultrafilter $U$ on a set $I$ is principal iff there is some $i^* \in I$ such that the set $\{ x \subseteq I \mid i^* \in x \}$ is in $U$.

   Let $\mathcal{L}$ be a language and $\langle \mathfrak{A}_i \mid i \in I \rangle$ be an indexed system of $\mathcal{L}$-structures. Assume that $U$ is a principal ultrafilter on $I$. Prove that the ultraproduct $\text{Ult}(\langle \mathfrak{A}_i \mid i \in I \rangle, U)$ is isomorphic to one of the structures $\mathfrak{A}_i$. Which one is it?

   Conclusion: In order that the ultrapower construction gives something new we need that the corresponding ultrafilter is non-principal.

3. Any ultrafilter is **finitely complete**, that is, if $X \subseteq U$ is finite then $\bigcap X \in U$. An ultrafilter $U$ on a set $I$ is **countably complete** if and only if for every countable $X \subseteq U$ we have $\bigcap X \in U$.

   Assume $\mathcal{L}$ is a language that contains (among other possible symbols) a binary predicate symbol $\dot{E}$. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure such that $\dot{E}^\mathfrak{A}$ is a well-founded relation. Let $U$ be an ultrafilter on $I$ and $\mathfrak{A}' = \text{Ult}(\mathfrak{A}, U)$. Prove:

   \[ \dot{E}^\mathfrak{A}' \text{ is well-founded } \iff \text{U is countably complete.} \]

   **Hint.** Using the Łoś Theorem reduce the question of well-foundedness of $\dot{E}^\mathfrak{A}'$ to that of well-foundedness of $\dot{E}^\mathfrak{A}$. Start with the implication $\iff$ which is simpler and which will give you a hint how to approach the converse.

4. Assume AC. Prove the compactness theorem using ultrapowers.

   **Hint.** Assume $\Sigma$ is a set of $\mathcal{L}$-sentences such that every finite $\Delta \subseteq \Sigma$ has a model. Your index set $I$ will be the set of all such $\Delta$‘s, i.e. $I = [\Sigma]<\omega$. To each $\Delta \in I$ pick an $\mathcal{L}$-structure $\mathfrak{A}_\Delta$ such that $\mathfrak{A}_\Delta = \Delta$. This can be done by the assumptions. Now for each $\Delta \in I$ let $X_\Delta = \{ \Delta' \in I \mid \Delta \subseteq \Delta' \}$. Show that there is an ultrafilter $U$ on $I$ such that $X_\Delta 
subseteq U$ for each $\Delta \in I$. Then show that $\text{Ult}(\langle \mathfrak{A}_\Delta \mid \Delta \in I \rangle, U) = \Sigma$. 

1
5. Let $\mathcal{N}$ be the standard model of arithmetic; here we understand that the corresponding language $\mathcal{L}$ contains the usual symbols $0, S, +, \cdot$ and $\prec$. Let $U$ be a nonprincipal ultrafilter on $\omega$ and let $\mathcal{N}' = \text{Ult}(\mathcal{N}, U)$.

Prove that real numbers can be embedded into $\mathcal{N}'$ in the following sense: There is a map $f : \mathbb{R} \to \mathcal{N}'$ such that for every $a, b \in \mathbb{R}$ we have

$$a < b \implies f(a) <_{\mathcal{N}'} f(b).$$

Here $<$ is the usual ordering on real numbers.

This says that the ultrapower $\mathcal{N}'$ has large cardinality and its ordering $<_{\mathcal{N}'}$ is complicated.