HOMEWORK 7

1. Assume AC. Let $\mathcal{L}$ be a countable language and $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure. Show that there is a club $\mathcal{C}$ in $\mathcal{P}(A)$ in the sense of Woodin's definition such that every $\bar{A} \in \mathcal{C}$ induces an elementary substructure of $\mathfrak{A}$.

Hint. Consider Skolem functions for $\mathfrak{A}$.

For all problems concerning ultrapowers assume that the structures in question have Skolem functions.

2. Recall that an ultrafilter U on a set I is principal iff there is some $i^* \in I$ such that the set $\{x \subseteq I \mid i^* \in x\}$ is in U.

Let $\mathcal{L}$ be a language and $\langle \mathfrak{A}_i \mid i \in I \rangle$ be an indexed system of $\mathcal{L}$-structures. Assume that U is a principal ultrafilter on I. Prove that the ultraproduct $\mathrm{Ult}(\langle \mathfrak{A}_i \mid i \in I \rangle, U)$ is isomorphic to one of the structures $\mathfrak{A}_i$. Which one is it?

Conclusion: In order that the ultrapower construction gives something new we need that the corresponding ultrafilter is non-principal.

3. Any ultrafilter is **finitely complete**, that is, if $X \subseteq U$ is finite then $\bigcap X \in U$. An ultrafilter U on a set I is **countably complete** if and only if for every countable $X \subseteq U$ we have $\bigcap X \in U$.

Assume $\mathcal{L}$ is a language that contains (among other possible symbols) a binary predicate symbol $\dot{E}$. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure such that $\dot{E}^{\mathfrak{A}}$ is a well-founded relation. Let U be an ultrafilter on I and $\mathfrak{A}' = \mathrm{Ult}(\mathfrak{A}, U)$. Prove:

$$\dot{E}^{\mathfrak{A}'} \text{ is well-founded} \iff U \text{ is countably complete.}$$

Hint. Using the Łoś Theorem reduce the question of well-foundedness of $\dot{E}^{\mathfrak{A}'}$ to that of well-foundedness of $\dot{E}^{\mathfrak{A}}$. Start with the implication $\Longleftarrow$ which is simpler and which will give you a hint how to approach the converse.

4. Assume AC. Prove the compactness theorem using ultrapowers.

Hint. Assume Σ is a set of $\mathcal{L}$-sentences such that every finite $\Delta \subseteq \Sigma$ has a model. Your index set I will be the set of all such Δ's, i.e. $I = [\Sigma]^{<\omega}$. To each $\Delta \in I$ pick an $\mathcal{L}$-structure $\mathfrak{A}_\Delta$ such that $\mathfrak{A}_\Delta \models \Delta$. This can be done by the assumptions. Now for each $\Delta \in I$ let $X_\Delta = \{\Delta' \in I \mid \Delta \subseteq \Delta'\}$. Show that there is an ultrafilter U on I such that $X_\Delta \in U$ for each $\Delta \in I$. Then show that $\mathrm{Ult}(\langle \mathfrak{A}_\delta \mid \Delta \in I \rangle, U) \models \Sigma$.

5. Let $\mathfrak{N}$ be the standard model of arithmetic; here we understand that the corresponding language $\mathcal{L}$ contains the usual symbols $\dot{0}, \dot{S}, \dot{+}, \dot{\cdot}$ and $\dot{<}$. Let U be a nonprincipal ultrafilter on ω and let $\mathfrak{N}' = \mathrm{Ult}(\mathfrak{N}, U)$.

Prove that real numbers can be embedded into $\mathfrak{N}'$ in the following sense: There is a map $f : \mathbb{R} \to \mathfrak{N}'$ such that for every $a, b \in \mathbb{R}$ we have

$$a < b \implies f(a) \dot{<}^{\mathfrak{N}'} f(b).$$

Here $<$ is the usual ordering on real numbers.

This says that the ultrapower $\mathfrak{N}'$ has large cardinality and its ordering $\dot{<}^{\mathfrak{N}'}$ is complicated.