HOMEWORK 1

Important point: It is important that you include all **relevant** details in your solutions. It is a part of the course to learn to recognize which details are relevant and which not.

1. Work in ZF. Give a "classical" proof of the implication

$$\mathrm{AC} \Longrightarrow \mathrm{Zorn's Lemma}.$$

That is, given a partial ordering $\mathbb{P} = \langle P, \leq \rangle$ satisfying Zorn's condition and some $q \in P$, find a maximal element p in $\mathbb{P}$ above q as follows.

Use the following notation. If $b \subseteq P$ then

$$A_b = \{z \in P \,|\, (\forall u \in b)(u \lneqq z)\}.$$

Fix a selctor f on $\mathcal{P}(P)$. Define

$\mathcal{F} =$ the set of all chains b in $\mathbb{P}$ such that

(a) q is the least element of b;

(b) for every proper initial segment c of b we have

$$f(A_c) \in b \text{ and if } z \in b \cap A_c \text{ then } f(A_c) \leq z.$$

Then proceed as follows:

(a) Show that if $b, b' \in \mathcal{F}$ then b is an initial segment of b' or vice versa.

(b) $b^* = \bigcup \mathcal{F}$ is in $\mathcal{F}$ and b^* has a largest element, call it p.

(c) p is a maximal element in $\mathbb{P}$ and $q \leq p$.

2. Give a direct proof that AC implies the well-ordering principle. Proceed similarly as in the proof of Zorn's lemma in the lecture. Let A be a set. You intend to find a well-ordering on A. Fix a selector f on $\mathcal{P}(A)$ and some element $c \notin A$. Then define, using recursion on ordinals, a map $F : \mathbf{On} \to A \cup \{c\}$ so that $F \restriction \alpha$ is injective whenever $F[\alpha] \subseteq A$ and $F[\alpha] = A$ for some $\alpha \in \mathbf{On}$. Give the definition of F and prove all its relevant properties.

3. Let V be a vector space over $\mathbb{R}$. (Here $\mathbb{R}$ is not relevant, any field would be OK.) A set $X \subseteq V$ is

(a) linearly independent iff for all $n \in \mathbb{N}$, all $x_1, \ldots, x_\ell \in X$ and all $a_1, \ldots, a_\ell \in \mathbb{R}$ we have

$$a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell = 0 \implies a_1 = a_2 = \cdots = a_\ell = 0$$

(b) generates the space V iff every element of V can be expressed as the sum $a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell$ for some $n \in \mathbb{N}$, $x_1, \ldots, x_\ell \in X$ and $a_1, \ldots, a_\ell \in \mathbb{R}$.

A set $X \subseteq V$ is a base of the space V iff X is both linearly independent and generates X.

Show that under AC, any linearly independent $X \subseteq V$ can be extended a base. That is, there is a base of V, say B such that $X \subseteq B$.

Hint. Use Zorn's lemma. Consider $\mathbb{P}$ to be the set of all linearly independent subsets of V containing X, partially ordered under inclusion.

4. Let $\mathbb{B}$ be a Boolean algebra. A maximal filter on $\mathbb{B}$ is called an ultrafilter. Prove

(a) A filter F is an ultrafilter iff for every $b \in \mathbb{B}$ either $b \in F$ or $b' \in F$. Recall that b' is the complement to b in $\mathbb{B}$.

(b) Use Zorn's lemma to show that if F is a filter on $\mathbb{B}$ then F can be extended to an ultrafilter, i.e. there is an ultrafilter U on $\mathbb{B}$ such that $F \subseteq U$.

Hint. Regarding (b), consider the poset $\mathbb{P}$ of all filters on $\mathbb{B}$ extending F, partially ordered under inclusion.