HOMEWORK 2

Important point: It is important that you include all **relevant** details in your solutions. It is a part of the course to learn to recognize which details are relevant and which not.

1. Work in ZF. Prove:

(a) AC $\Longrightarrow$ DC.

(b) For every set A we have $\mathrm{DC}(A) \Longrightarrow \mathrm{AC}_\omega(A)$

(c) For every set A, the axiom $\mathrm{DC}(A)$ is equivalent to the statement: If $T \subseteq {}^{<\omega}A$ is a tree such that

- Every $s \in T$ has a proper lengthening in T

then T has an infinite branch.

Hint. Recall the material on weak versions of the Axiom of Choice. This also included the material on trees used in (c). In (b) you have to come up with the relation that will return a sequence of elements of A needed for $\mathrm{AC}_\omega(A)$. In (c) you have to prove two implications. To get the existence of an infinite branch through T you have to come up with a binary relation that together with $\mathrm{DC}(A)$ will give you the branch. In order to get $\mathrm{DC}(A)$ you have to find a suitable tree $T \subseteq {}^{<\omega}A$. Its branch will be the sequence as in $\mathrm{DC}(A)$.

2. Work in ZF. Cantor's theorem tells us that there is no surjection of A onto $\mathcal{P}(A)$, and so there is no injection from $\mathcal{P}(A)$ into A. Prove the following version of Cantor's theorem for injections: There is no injection of $\mathcal{P}^*(A)$ into A where

$$\mathcal{P}^*(A) = \{x \in \mathcal{P}(A) \mid x \text{ is well-orderable}\}.$$

Notice that in the absence of the Axiom of Choice this is much stronger than the statement "there is no injection from $\mathcal{P}(A)$ into A", as there may be only very few well-orderable subsets of A. This exercise actually shows that even in the absence of the Axiom of Choice, there are still many well-orderable subsets of A.

Hint. Proceed by contradiction. Assume there is such an injection, say $f : \mathcal{P}^*(A) \to A$. Fix some $c \notin A$. By recursion on ordinals define a map

$F : \mathbf{On} \to A \cup \{c\}$ such that $F(\alpha) = f(F[\alpha])$ whenever $F[\alpha] \subseteq A$. Then show that $F \restriction \alpha$ is injective whenever $F[\alpha] \subseteq A$ and obtain a contradiction. Write down all relevant details carefully.

3. Work in ZFC.

(a) Let $\langle \kappa_i \,|\, i \in I \rangle$ be a system of cardinals, all $\kappa_i \neq 0$. Show that

$$\sum_{i \in I} \kappa_i = (\sup_{i \in I} \kappa_i) \cdot |\, I \,|\,.$$

(b) Let λ be an infinite cardinal. Show that λ can be split into λ many disjoint sets of size λ. That is, show that there are sets $A_\xi \subseteq \lambda$ for $\xi < \lambda$ that are mutually disjoint and

$$\bigcup_{\xi < \lambda} A_\xi = \lambda.$$

Give an example of such a decomposition.

(c) Let $\langle \kappa_i \,|\, i \in I \rangle$ be a system of cardinals, all $\kappa_i \neq 0$. Let $\langle A_j \,|\, j \in J \rangle$ be a partition of I. Construct a bijection witnessing that

$$\prod_{i \in I} \kappa_i \sim \prod_{j \in J} \left(\prod_{i \in I} \kappa_i \right).$$

This can be viewed as the combination of the associativity and commutativity of multiplication.

(d) Let $\langle \kappa_\xi \,|\, \xi < \lambda \rangle$ be an increasing, **but not necessarily strictly**, sequence of cardinals where λ is a cardinal. Assume all $\kappa_\xi \neq 0$. Let $\kappa = \sup_{\xi < \lambda} \kappa_\xi$. Show that

$$\prod_{\xi < \lambda} \kappa_i = \kappa^\lambda.$$

Hint. Regarding (a): Prove two inequalities. Both require just straightforward computations with supremas (getting useful upper bounds) and knowledge about products of two cardinals. Regarding (b): Recall that $\lambda \times \lambda$ is equinumerous to λ. Regarding (c): If p is a map with domain I, look at all restriction of p on the sets A_j. Regarding (d): Prove two inequalities. $\leq$ is easy. To prove $\geq$ use (b) and (c). Neither of the inequalities requires a direct use of the Schröder-Bernstein theorem.