HOMEWORK 3

Important point: It is important that you include all **relevant** details in your solutions. It is a part of the course to learn to recognize which details are relevant and which not.

1. Work in ZFC. Let κ be an infinite cardinal. Prove that the following two facts are equivalent.

(a) $\mathrm{cf}(\kappa) \leq \gamma$.

(b) There is a family of sets $\langle A_\xi \mid \xi < \gamma \rangle$ such that each A_ξ has cardinality strictly smaller than κ and

$$\kappa = \bigcup_{\xi<\gamma} A_\xi.$$

Conclude that the least ordinal γ with the property

- κ can be expressed as the union of a family $\langle A_\xi \mid \xi < \gamma \rangle$ where each A_ξ has size strictly smaller than κ

is equal to $\mathrm{cf}(\kappa)$.

Hint. Show that if κ is the union of sets A_ξ for $\xi < \gamma$ where $|A_\xi| < \kappa$ then the cardinals $\kappa_\xi = |A_\xi|$ are unbounded in κ.

2. Work in ZFC. Let κ be an infinite cardinal and $\lambda \leq \kappa$ be a cardinal. Recall that

$$[\kappa]^\lambda = \{x \subseteq \kappa \mid \ |x| = \lambda\}.$$

Prove that $|[\kappa]^\lambda| = \kappa^\lambda$.

Hint. Prove two inequalities. Here use the Schröder-Bernstein Theorem. To see $\leq$, first show that for each $\delta \in [\lambda, \lambda^+)$, each subset of order-type δ can be identified with a function from δ into κ. This gives you an injection from $[\kappa]^\lambda$ into $\bigcup\{{}^\delta\kappa \mid \delta \in [\lambda, \lambda^+)\}$. To see $\geq$, show that each function from λ to κ can be identified with a subset of $\kappa \times \lambda$.

3. Work in ZF. Let κ be an infinite cardinal. Define the class H_κ as follows:

$$x \in H_\kappa \iff |\operatorname{trcl}(x)| < \kappa.$$

(a) Prove that $H_\kappa \subseteq V_\kappa$. Conclude that H_κ is a set.

(b) Prove that H_κ is transitive.

(c) Prove that $H_\omega = V_\omega$.

(d) Work in ZFC. Prove that if κ is strongly inaccessible then $V_\kappa = H_\kappa$.

(f) Work in ZFC. Prove that if κ is a successor cardinal then H_κ is a proper subset of V_κ.

Hint. Regarding (a): Inspect the rank function restricted to $\mathrm{trcl}(\{x\})$ for each $x \in H_\kappa$. Regarding (b) and (c): This is easy. Regarding (d): By induction on $\alpha < \kappa$ show that the size of V_α is strictly smaller than κ. Regarding (e): use Cantor's theorem.