HOMEWORK 3

Important point: It is important that you include all relevant details in your solutions. It is a part of the course to learn to recognize which details are relevant and which not.

1. Work in ZFC. Let \( \kappa \) be an infinite cardinal. Prove that the following two facts are equivalent.

(a) \( \text{cf}(\kappa) \leq \gamma. \)

(b) There is a family of sets \( \langle A_\xi \mid \xi < \gamma \rangle \) such that each \( A_\xi \) has cardinality strictly smaller than \( \kappa \) and

\[
\kappa = \bigcup_{\xi < \gamma} A_\xi.
\]

Conclude that the least ordinal \( \gamma \) with the property

- \( \kappa \) can be expressed as the union of a family \( \langle A_\xi \mid \xi < \gamma \rangle \) where each \( A_\xi \) has size strictly smaller than \( \kappa \)

is equal to \( \text{cf}(\kappa) \).

**Hint.** Show that if \( \kappa \) is the union of sets \( A_\xi \) for \( \xi < \gamma \) where \( |A_\xi| < \kappa \) then the cardinals \( \kappa_\xi = |A_\xi| \) are unbounded in \( \kappa \).

2. Work in ZFC. Let \( \kappa \) be an infinite cardinal and \( \lambda \leq \kappa \) be a cardinal. Recall that

\[
[\kappa]^\lambda = \{ x \subseteq \kappa \mid |x| = \lambda \}.
\]

Prove that \( |[\kappa]^\lambda| = \kappa^\lambda. \)

**Hint.** Prove two inequalities. Here use the Schröder-Bernstein Theorem. To see \( \leq \), first show that for each \( \delta \in [\lambda, \lambda^+) \), each subset of order-type \( \delta \) can be identified with a function from \( \delta \) into \( \kappa \). This gives you an injection from \( [\kappa]^\lambda \) into \( \bigcup \{ \delta \kappa \mid \delta \in [\lambda, \lambda^+) \} \). To see \( \geq \), show that each function from \( \lambda \) to \( \kappa \) can be identified with a subset of \( \kappa \times \lambda \).

3. Work in ZF. Let \( \kappa \) be an infinite cardinal. Define the class \( H_\kappa \) as follows:

\[
x \in H_\kappa \iff |\text{trcl}(x)| < \kappa.
\]
(a) Prove that $H_\kappa \subseteq V_\kappa$. Conclude that $H_\kappa$ is a set.

(b) Prove that $H_\kappa$ is transitive.

(c) Prove that $H_\omega = V_\omega$.

(d) Work in ZFC. Prove that if $\kappa$ is strongly inaccessible then $V_\kappa = H_\kappa$.

(f) Work in ZFC. Prove that if $\kappa$ is a successor cardinal then $H_\kappa$ is a proper subset of $V_\kappa$.

**Hint.** Regarding (a): Inspect the rank function restricted to $\text{trcl}(\{x\})$ for each $x \in H_\kappa$. Regarding (b) and (c): This is easy. Regarding (d): By induction on $\alpha < \kappa$ show that the size of $V_\alpha$ is strictly smaller than $\kappa$. Regarding (e): use Cantor’s theorem.