MATH 281A FALL 2019 HOMEWORK 2

Target date: Wednesday, November 6, 2019

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Let κ be a regular cardinal and F be a uniform normal filter on κ.

(a) Prove that F is κ-complete.

(b) Prove that $\mathsf{CLUB}_\kappa \subseteq F$, that is, CLUB_κ is the smallest uniform normal ideal on κ.

2. (1/2 page) Let κ be a regular cardinal and $S \subseteq \kappa$. Recall that an ordinal α is a limit point of S iff $\sup(S \cap \alpha) = \alpha$. Also recall that if S is stationary then an ordinal $\alpha < \kappa$ is a reflection point of S iff $S \cap \alpha$ is stationary in α. Thus, every reflection point of S is a limit point of S of uncountable cofinality. Define

$$\mathsf{succ}(S) = \{\alpha \in S \mid \alpha \text{ is not a limit point of } S\}$$

and

$$\mathsf{N}(S) = \{\alpha \in S \mid \alpha \text{ is not a reflection point of } S\}$$

Prove that if S is a stationary subset of κ then $\mathsf{succ}(S) \in \mathsf{NS}_\kappa$ and $\mathsf{N}(S)$ is a stationary subset of κ.

3. (1/2 page) Recall that we consider ordinals as topological spaces with the interval topology, and similarly the structure $\mathbb{R}$ of all real numbers also as a topological space with the interval topology. Prove the following

(a) Every continuous map $f : \omega_1 + 1 \to \mathbb{R}$ is eventually constant.

(b) Every continuous map $f : \omega_1 \to \mathbb{R}$ is eventually constant.

Clause (a) follows easily, but I am including it for a comparison with clause (b).

4. (1/3 page) Consider a universe where ω_1 is an everyday reality. Assume there is a train network with $\omega_1 + 1$ many stations which are indexed by ordinals below $\omega_1 + 1$, say $(s_\alpha \mid \alpha < \omega_1 + 1)$ is a list of stations such that the trains depart from station s_0 and stop at each s_α in the increasing order. The last station is s_{ω_1}.

Now consider the following situation. At each station s_α one passenger gets off, if there is any on the train, and ω many get on. It is understood that the passenger who gets off does not reenter.

Question: How many passengers arrive at station s_{ω_1}?

5. (1/2 page) Let α, β be ordinals of uncountable cofinality and $f : \alpha \to \beta$ be a normal cofinal map. Prove the following.

(a) If $S \subseteq \alpha$ is a stationary subset of α then $f[S]$ is a stationary subset of β.
(b) If $T \subseteq \beta$ is a stationary subset of β then $f^{-1}[T]$ is a stationary subset of α.