MATH 281A FALL 2019 HOMEWORK 3

Target date: Wednesday, November 27, 2019

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Let $\mu < \kappa$ be regular cardinals. Define

$$E^\kappa_\mu = \{\alpha < \kappa \mid \mathsf{cf}(\alpha) = \mu\}.$$

(a) Prove that if λ is a regular cardinal such that $\mu < \lambda < \kappa$ then the the set of all reflection points of E^κ_μ is unbounded in κ and closed under limits of cofinality λ.

(b) Assume $\kappa = \mu^+$. How big is the set of all reflection points of E^κ_μ?

2. (1/2 page) Let κ be a regular cardinal, A be a set of cardinality larger than κ and C be a club in $[A]^{<\kappa}$ in the sense of Jech. Let θ be a cardinal such that $C \in H_\theta$ and $x \in \mathcal{P}_\kappa(H_\theta)$ be an elementary substructure of H_θ such that $C \in x$. Prove that $x \cap A \in C$.

3. (2/3) page Let $\mu < \kappa$ be cardinals and κ be regular. Assume T is a tree of height κ such that each level of T has cardinality $< \mu$. Prove that T has a cofinal branch.

4. (1/2) page Let κ be regular and T be a tree of height κ.

(a) Assume there is a split above every node of T. Prove that if T has a cofinal branch than T has an antichain of cardinality κ.

(b) Prove that if T is Aronszajn and pruned then there is a split above every node of T.

(c) Assume κ is a successor cardinal and T is special. Prove that T is not a Suslin tree.

5. (1/2 page) Let $(L, <)$ be a Suslin line. Recall that we defined an equivalence relation $\sim$ on L as follows:

$$x \sim y \iff \text{the open interval } (x, y) \text{ is separable.}$$

Prove that each equivalence class $[x]$ is a separable interval in $(L, <)$. A singleton is considered an interval here.

6. (1 page) Let κ be a cardinal.

(a) Prove that if $\diamondsuit_\kappa$ holds then κ is regular.

(b) Prove that $\diamondsuit_\kappa$ implies the equality $2^{<\kappa} = \kappa$, so in particular if $\kappa = \mu^+$ we have $2^\mu = \kappa = \mu^+$ and $\diamondsuit \Rightarrow \mathsf{CH}$.

(c) If $\diamondsuit_\kappa(S)$ holds then there is a family $\mathcal{F}$ of stationary subsets of S such that

   (i) $\mathsf{card}(\mathcal{F}) = 2^\kappa$, and

   (ii) If $S_1, S_2 \in \mathcal{F}$ are such that $S_1 \neq S_2$ then $S_1 \cap S_2$ is bounded in κ.

That is, $\mathsf{NS}_\kappa \restriction S$ is as non-saturated as it can possibly be.

(d) Prove that if the definition of a $\diamondsuit^*_\kappa$-sequence $(A_\alpha \mid \alpha < \kappa)$ is strengthened in that the requirement

– $\mathsf{card}(A_\alpha) \leq \mathsf{card}(\alpha)$

is strengthened to

– $\mathsf{card}(A_\alpha) = 1$

then this strengthened variant of the $\diamondsuit^*_\kappa$-principle is inconsistent. Prove that even the weaker strengthening where we request

– $\mathsf{card}(A_\alpha) < \mu$

for some fixed $\mu < \kappa$ is inconsistent.