MATH 281A FALL 2019 HOMEWORK 4

Target date: Tuesday, December 17, 2019

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) Construct an Aronszajn tree using injections $t : \alpha \to \mathbb{Q}$, where $\alpha < \omega_1$, as nodes. The tree ordering is again the inclusion.

2. (1/2 page) Prove that there is no order-preserving map

$$\sigma : (T, <_T) \to (\mathbb{R}, <)$$

for any Suslin tree $(T, <_T)$. Here $<$ is the natural ordering of real numbers. Recall that a map $\sigma : (T, <_T) \to (\mathbb{R}, <)$ is order-preserving iff

$$s <_T t \implies \sigma(s) < \sigma(t)$$

for all $s, t \in T$.

3. (2/3 page) Assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Fix an enumeration $(a_\alpha \mid \alpha < \omega_2)$ of all bounded subsets of ω_2 such that each bounded subsets of ω_2 occurs unboundedly many times on this enumeration. For each $\alpha \in E^{\omega_2}_\omega$ define

$$A_\alpha = \{\bigcup_{n\in\omega} a_{\alpha_n} \mid (\alpha_n \mid n \in \omega) \text{ is a sequence cofinal in } \alpha\} \cap \mathcal{P}(\alpha)$$

Prove that $(A_\alpha \mid \alpha \in E^{\omega_2}_\omega)$ is a $\diamondsuit'_{\omega_2}(E^{\omega_2}_\omega)$-sequence.

More challenging task for those who are interested: Prove that for every stationary set $S \subseteq E^{\omega_2}_\omega$, the sequence $(A_\alpha \mid \alpha \in S)$ is a $\diamondsuit_{\omega_2}(S)$-sequence.

4. (1/2 page) Let λ be a regular cardinal.

(a) Assume $S \subseteq \lambda$ is stationary and $(c_\alpha \mid \alpha < \lambda)$ is a $\square(\lambda, S)$-sequence. Prove that $(c_\alpha \mid \alpha < \lambda)$ does not have a thread.

(b) Assume $\lambda = \kappa^+$ and $\square_\kappa$ holds. Prove that for every stationary $S \subseteq \lambda \cap E^\lambda_\omega$ there is a stationary $T \subseteq S$ such that $\square(\lambda, T)$ holds (hence T is non-reflecting).

It should be stressed that a thread of a coherent sequence of length λ must be a cofinal subset of λ, otherwise the notion of a thread would trivialize. I am not sure if I said this when I gave the official definition in the lecture, but I am emphasizing it now in order to avoid misunderstandings.

5. (1/3 page) Prove that $\mathsf{MA}(\omega_1)$ implies there is no Suslin tree.