MATH 281B WINTER 2020 HOMEWORK 1

Target date: Thursday, February 6, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1 page) Recall the poset we used in the lecture to code subsets of κ by subsets of ω. The conditions had the form $\langle s, F\rangle$ where s was a finite subset of ω and F was a finite subcollection of a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$. This forcing is in the literature also called **almost disjoint forcing**.

Given two infinite sequences $a, b \in {}^{\omega}\omega$, write $a <^* b$ iff there is some $n \in \omega$ such that

$$a(i) < b(i) \quad \text{for all } i > n.$$

We also say that b eventually dominates a.

(a) Prove in ZFC that there is a sequence $\langle a_\xi \mid \xi < \omega_1\rangle$ such that $a_\xi < a_{\xi'}$ whenever $\xi < \xi'$.

(b) Prove that $\mathsf{MA}(\kappa)$ implies there is a sequence $\langle a_\xi \mid \xi < \kappa^+\rangle$ such that $a_\xi < a_{\xi'}$ whenever $\xi < \xi'$.

For (b), use ideas we discussed in connection with almost disjoint forcing.

2. (1 page) Assume Γ, Δ are classes of posets such that for every $\mathbb{P} \in \Gamma$ there is some $\mathbb{Q} \in \Delta$ such that $\mathbb{Q}$ is a dense subset of $\mathbb{P}$.

(A) Prove that $\mathsf{FA}(\Delta, \kappa) \implies \mathsf{FA}(\Gamma, \kappa)$.

Now assume that every $\mathbb{Q} \in \Delta$ is a dense subset of some $\mathbb{P} \in \Gamma$.

(B) Prove that $\mathsf{FA}(\Gamma, \kappa) \implies \mathsf{FA}(\Delta, \kappa)$.

Use (A) and (B) to prove the equivalence of the following three statements.

(a) $\mathsf{MA}(\kappa)$.

(b) If $\mathbb{B}$ is a complete c.c.c. Boolean algebra and $\mathcal{D}$ is a family of dense subsets of $\mathbb{B}$ with $\mathsf{card}(\mathcal{D}) \le \kappa$ then there is a $\mathcal{D}$-generic ultrafilter on $\mathbb{B}$.

(c) If X is a compact Hausdorff c.c.c. topological space and $\mathcal{D}$ is a family of dense open subsets of X with $\mathsf{card}(\mathcal{D}) \le \kappa$ then $\bigcap \mathcal{D}$ is a dense subset of X.

3. (2/3 page) Show that $\mathsf{MA}(\kappa)$ is a statement about $\mathcal{P}(\kappa)$, in the following sense: Prove that

- $\mathsf{MA}(\kappa) \iff \mathsf{FA}(\Gamma, \kappa)$

where Γ is the class of all c.c.c. posets which have cardinality at most κ.

4. (1/2 page) Work in ZF. Assume there is an ordinal α such that $V_\alpha \models \mathsf{ZF}$. Prove that the smallest such α has cofinality ω.

5. (1/3 page) Assume M, N are transitive (may be set or proper class) models of ZF and α is an ordinal such that $\alpha \in M \cap N$. Assume $\mathcal{P}(\alpha)^M = \mathcal{P}(\alpha)^N$. Prove that $\alpha^{+M} = \alpha^{+N}$.

If a model M is a set, we allow the option that $\alpha^{+M} = \mathbf{On} \cap M$, so $\alpha^{+M} \notin M$ in this case.