MATH 281B WINTER 2020 HOMEWORK 2

Target date: Tuesday, February 25, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not reprove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (2/3 page) Assume M, N are two transitive proper class models of ZF such that for every ordinal α we have $\mathcal{P}(\alpha)^M = \mathcal{P}(\alpha)^N$.

(a) Assume both M, N satisfies AC. Prove that $M = N$.

(b) Now assume one of M, N satisfies AC. Prove that still $M = N$.

Notice that for the conclusions in (a),(b) we actually do not need that M, N are classes, that is, they need not have to be definable.

2. (3/4 page) Recall that if M, N are structures for LST_A then a map $\sigma : M \to N$ is said to be Σ_k-preserving iff for every Σ_k-formula $\varphi(v_1, \ldots, v_\ell)$ we have

$$M \models \varphi(a_1, \ldots, a_\ell) \iff N \models \varphi(\sigma(a_1), \ldots, \sigma(a_\ell)) \tag{1}$$

whenever $a_1, \ldots, a_\ell \in M$. In what follows we will write A_M to denote the interpretation $\dot{A}^M$ of the unary relation symbol $\dot{A}$, and analogously for A_N.

We say that σ is Q-preserving iff (1) holds for every Q-formula $\varphi(v_1, \ldots, v_\ell)$ and any $a_1, \ldots, a_\ell \in M$.

Finally if M, N are transitive then we say that σ is **cofinal** iff

$$N = \bigcup \mathrm{rng}(\sigma),$$

or equivalently every $y \in N$ is an element of $\sigma(x)$ for some $x \in M$.

Assuming M, N are transitive, prove the following.

(a) If σ is Σ_k-preserving and $\varphi(v_1, \ldots, v_\ell)$ is a Σ_{k+1}-formula then for every $a_1, \ldots, a_\ell \in M$ we have

$$M \models \varphi(a_1, \ldots, a_\ell) \implies N \models \varphi(\sigma(a_1), \ldots, \sigma(a_\ell))$$

(b) If σ is Σ_k-preserving and $\varphi(v_1, \ldots, v_\ell)$ is a Π_{k+1}-formula then for every $a_1, \ldots, a_\ell \in M$ we have

$$N \models \varphi(\sigma(a_1), \ldots, \sigma(a_\ell)) \implies M \models \varphi(a_1, \ldots, a_\ell)$$

(c) If σ is Σ_0-preserving and cofinal then σ is Σ_1-preserving.

(d) Assume for every $x \in M$ we have $\bigcup x \in M$. If σ is Σ_0-preserving and cofinal then σ is Q-preserving.

(e) Use (d) to show: If M is A_M-rudimentarily closed and σ is Σ_0-preserving and cofinal then N is A_N-rudimentarity closed.

3. (2/3 page) Assume M, N are transitive structures of LST, $M \models \mathsf{BST}$, $A \subseteq M$ is amenable to M and $\sigma : M \to N$ is a Σ_0-preserving with respect to LST and cofinal. Define $A' \subseteq N$ by

$$A' = \bigcup\{A \cap x \mid x \in M\}$$

(a) Prove that A' is amenable to N.
(b) Prove that if A'' is amenable to N and $\sigma : (M, A) \to (N, A'')$ is Σ_0-preserving with respect to LST_A then $A'' = A'$.
(c) Prove that $\sigma : (M, A) \to (N, A')$ is Σ_0-preserving with respect to A'.

4. (1/2 page) Assume M, N are transitive structures and $\sigma : M \to N$ is a Σ_0-preserving cofinal map. Assume $M \models \mathsf{ZF}^-$. Prove that σ is fully elementary. (So $N \models \mathsf{ZF}^-$ as well.)

5. (1 page) Let κ be regular and $\theta >> \kappa$. For a set $X \subseteq H_\theta$ we write $\alpha_X = \sup(X \cap \kappa)$. In what follows we are considering only sets $X \subseteq H_\theta$ for which $\alpha_X \in \kappa$. Prove the following.

We think of H_θ as a structure equipped with a well-ordering, and we add a symbol to LST for this well-ordering which we suppress in our notation. In this sense the structure H_θ has canonically definable Skolem functions, for instance, so it makes sense to talk about hulls. Also, elementary substructures of H_θ are understood to be with respect to this expanded language.

(a) If C is a club in κ and X is an elementary substructure of H_θ such that $C \in X$ then $\alpha_X \in C$.
(b) If $\mathcal{C} = \langle C_\xi \mid \xi < \kappa\rangle$ is a sequence of clubs in κ such that $\mathcal{C} \in X$ then $\alpha_X \in \triangle_{\xi<\kappa} C_\xi$.
(c) A set $S \subseteq \kappa$ is stationary in κ iff there is an elementary substructure X of H_θ such that $S \in X$ and $\alpha_X \in S$.
(d) If S is a stationary subset of κ then

$$S^* = \{X \in \mathcal{P}(H_\theta) \mid \ X \text{ is an elementary substructure of } H_\theta \text{ and } \alpha_X \in S\}$$

is a stationary subset of $\mathcal{P}(H_\theta)$.

Regarding (b), it is easier to prove the result if we assume $X \cap \kappa = \alpha_X$. So you may want to try this simpler version before you try the the version in (b). Also, notice that this simpler version can be used for a proof that the diagonal intersection $\triangle_{\xi<\kappa} C_\xi$ contains a club.

6. (1 page) Prove the following.

(a) If $\kappa > \omega$ is a regular cardinal in $\mathbf{L}$ then $J_\kappa \models \mathsf{ZFC}^-$.
(b) If $\kappa > \omega$ is a cardinal in $\mathbf{L}$ then $J_\kappa = H_\kappa^{\mathbf{L}}$.
(c) If $A \subset \omega_1^{\mathbf{L}[A]}$ then $\mathbf{L}[A] \models \mathsf{CH}$.
(d) Assume $\mathbf{V} \models 2^\omega = \omega_2$. Construct a set $A \subseteq \omega_2^{\mathbf{V}}$ such that
  (i) $\omega_1^{\mathbf{L}[A]} = \omega_1^{\mathbf{V}}$,
  (ii) $\omega_2^{\mathbf{L}[A]} = \omega_2^{\mathbf{V}}$, and
  (iii) $\mathbf{L}[A] \models 2^\omega = \omega_2$.

Notice that (b) can be used to prove the non-trivial part of (a). Notice also that (a) has a direct proof that does not use (b). Finally notice that $J_1 = H_\omega = V_\omega$.