MATH 281B WINTER 2020 HOMEWORK 3

Target date: Tuesday, March 24, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (2/3 page) Work in ZFC. Assume κ is regular and θ is a cardinal much larger than κ. We will consider the structure H_θ as introduced in Homework 2, Exercise 5. Recall that for $X \subseteq H_\theta$ we write $\alpha_X = \sup(\kappa \cap X)$.

Assume X is an elementary substructure of H_θ such that $\alpha_X < \kappa$ and let

$$Y = \mathsf{Hull}^{H_\theta}(X \cup \alpha_X).$$

Prove that $\alpha_Y = \alpha_X$.

The following few exercises focus on weak variants of Axiom of Choice; they follow from AC by straightforward arguments.

2. (1/2 page) Work in ZF. Let κ be an infinite cardinal and A be a set. Then $\mathsf{AC}_\kappa(A)$ is the following statement.

> If $\langle A_\xi \mid \xi < \kappa \rangle$ is a sequence of sets such that $\varnothing \neq A_\xi \subseteq A$ for all $\xi < \kappa$ then there is a function $f : \kappa \to A$ such that $f(\xi) \in A_\xi$ for all $\xi < \kappa$.

Let $\mathsf{AC}^-_\kappa(A)$ be the weakening of $\mathsf{AC}_\kappa(A)$ which postulates the existence of a function f as above only for those sequences $\langle A_\xi \mid \xi < \kappa \rangle$ which are pairwise disjoint.

Prove that $\mathsf{AC}^-_\kappa(\mathcal{P}(\kappa))$ implies that κ^+ is regular.

3. (3/2 page) Work in ZF. Let A be a set. Then $\mathsf{DC}(A)$ is the following statement.

> If R is a binary relation on A satisfying
> $$(\forall x)(\exists y)(\langle x, y \rangle \in R)$$
> then for every $a \in A$ there is a function $f : \omega \to A$ such that
> (i) $f(0) = a$, and
> (ii) $\langle f(i), f(i+1) \rangle \in R$ for all $i \in \omega$.

Here "DC" stands for "dependent choice", meaning that the choice of the element at stage n depends on the sequence chosen up to stage n.

We will also consider trees on a set A in the same sense as done in descriptive set theory, so a tree T on A is a subset of ${}^{<\omega}A$ which is closed under initial segments. The notion of an infinite branch through T is the same as in descriptive set theory.

Prove that the following holds for every set A.

(a) $\mathsf{DC}(A) \implies \mathsf{AC}^-_\omega(A)$.
(b) $\mathsf{DC}(\omega \times A) \implies \mathsf{AC}_\omega(A)$.
(c) If there is a surjection of A onto $\omega \times A$ then $\mathsf{DC}(A) \implies \mathsf{AC}_\omega(A)$.
(d) The following are equivalent:
  (i) $\mathsf{DC}({}^{<\omega}A)$
  (ii) If T is a tree on A such that every $s \in T$ has a proper extension in T then T has an infinite branch.
(e) Prove the following.
  (i) Clause (ii) in (d) above implies $\mathsf{DC}(A)$
  (ii) If there is a surjection of A onto B then $\mathsf{DC}(A) \implies \mathsf{DC}(B)$. Thus, if there is a surjection of A onto ${}^{<\omega}A$ then the converse to (i) holds, that is, $\mathsf{DC}(A)$ implies clause (ii) in (d) above.

  Notice that a sufficient condition for the existence of a surjection of A onto ${}^{<\omega}A$ is the existence of surjections of A onto $A \times A$ and of A onto $\omega \times A$.
(f) $\mathsf{AC}^-_\omega(\mathcal{N}) \iff \mathsf{AC}_\omega(\mathcal{N})$, so $\mathsf{DC}(\mathcal{N}) \implies \mathsf{AC}_\omega(\mathcal{N})$.
(g) The following are equivalent:
  (i) $\mathsf{DC}(\mathcal{N})$
  (ii) If T is a tree on $\mathcal{N}$ such that every $s \in T$ has a proper extension in T then T has an infinite branch.

4. (1 page) Work in ZF. The **Axiom of Dependent choices** DC is the statement

$$(\forall A)\mathsf{DC}(A).$$

Prove that the following are equivalent.
(a) DC.
(b) Let $\mathcal{L}$ be a first order language such that $\mathsf{card}(\mathcal{L}) \leq \omega$. If $\mathcal{M}$ is an $\mathcal{L}$-structure then $\mathcal{M}$ has a countable elementary substructure.

5. (1/3 page) Work in ZF. Assume M is a transitive model of ZF^- and $S, T \in M$ are such that T is a tree on $\omega \times \kappa$ and T is a tree on $\omega \times \lambda$ where $\kappa, \lambda \in \mathbf{On} \cap M$. Prove the following:

$$M \models p[S] \cap p[T] \neq \varnothing \quad \iff \quad \mathbf{V} \models p[S] \cap p[T] \neq \varnothing$$

(One direction is, of course, trivial.)