MATH 281C SPRING 2020 HOMEWORK 1

Target date: Tuesday, May 12, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Work in ZF. Assume J^A_α is an acceptable structure and $\kappa \in J^A_\alpha$. Prove the following.

(a) If $\kappa > \omega$ a regular cardinal in the sense of J^A_α then J^A_κ is a model of ZFC^-.
(a) If κ is a singular cardinal in the sense of J^A_α then J^A_κ is a model of Zermelo Set Theory with Axiom of Choice. (So the only axiom of ZFC which may possibly fail is Collection.)

2. (1 page) Work in $\mathbf{L}$. Assume κ is a cardinal and

$$C = \{\tau \in (\kappa, \kappa^+) \mid J_\tau \text{ is an elementary substructure of } J_{\kappa^+}\}$$

Given a $\tau \in C$, let

$$\beta(\tau) = \text{the unique } \beta \text{ such that } \tau \text{ is a cardinal in } J_\beta \text{ but not in } J_{\beta+1}.$$

Notice that for $\tau \in C$ we have $\beta(\tau) > \tau$. We write briefly ϱ^n_β for $\varrho^n_{J_\beta}$. Prove the following.

(a) If $\kappa < \beta < \kappa^+$ and $\varrho^1_\beta \leq \kappa$ then $\varrho^1_\beta = \kappa$.
(b) There is a stationary set $S \subseteq C$ such that $\varrho^1_{\beta(\tau)} = \kappa$ for all $\tau \in S$.
(c) There is a club $C' \subseteq C$ such that for every $\tau \in C'$ there is some β with $\varrho^1_\beta = \tau$. (Why is this not a contradiction with (b)?)
(d) Assume $\langle \tau_n \mid n \in \omega \rangle$ is a strictly increasing sequence of ordinals in C such that $\varrho^1_{\beta(\tau_n)} = \kappa$. Let $\tau = \sup_{n\in\omega} \tau_n$. Is it true that $\varrho^1_{\beta(\tau)} = \kappa$?

The next exercises are on direct limits of transitive structures for LST_A.

Throughout the rest of the exercises, work in ZF.

Here we focus on well-ordered direct limits, which are the simplest ones. The common setting for the following is a commutative diagram

$$\langle (M_\xi, A_\xi), \pi_{\bar\xi,\xi} \mid \bar\xi \leq \xi < \lambda \rangle$$

where λ is a limit ordinal, each (M_ξ, A_ξ) is a transitive amenable structure, and each $\pi_{\bar\xi,\xi} : (M_{\bar\xi}, A_{\bar\xi}) \to (M_\xi, A_\xi)$ is a Σ_0-preserving map. We also assume that $\pi_{\xi,\xi} = \mathsf{id} \restriction M_\xi$ for all $\xi < \lambda$.

We let
$$D_0 = \{\langle \xi, x\rangle \mid x \in M_\xi\}$$
and on D_0 we introduce a binary relation I by
$$\langle \xi, x\rangle \; I \; \langle \xi', x'\rangle \iff (\xi \le \xi' \wedge \pi_{\xi,\xi'}(x) = x') \vee (\xi' \le \xi \wedge \pi_{\xi',\xi}(x') = x)$$

For the moment, assume the above commutative diagram is a set, i.e. it is an element of $\mathbf{V}$. In particular, all (M_ξ, A_ξ) are sets.

3. (use common sense regarding the length) Prove the following.

(a) I is an equivalence relation on D_0.

Let $D = D_0/I$ and write briefly $[\xi, x]$ to denote the equivalence class of $\langle \xi, x\rangle$. On D define a binary relation E and a unary relation A' by
$$[\xi, x] \; E \; [\xi', x'] \iff (\xi \le \xi' \wedge \pi_{\xi,\xi'}(x) \in x') \vee (\xi' \le \xi \wedge \pi_{\xi',\xi}(x') \in x)$$
and
$$A'([\xi, x]) \iff A_\xi(x)$$

(b) E, A' are well-defined, that is, their definitions do not depend on the choice of representatives.

Let
$$\mathbb{D} = (D, E, A')$$
be the LST_A-structure where the symbol for $\in$ is interpreted as E and the symbol $\dot{A}$ for a unary predicate is interpreted as A'. For $\xi < \lambda$ define maps
$$\pi'_\xi : (M_\xi, A_\xi) \to \mathbb{D}$$
by
$$\pi'_\xi(x) = [\xi, x]$$

(c) Prove that the maps π'_ξ are Σ_0-preserving and $\pi'_\xi = \pi'_{\xi'} \circ \pi_{\xi,\xi'}$ whenever $\xi < \xi'$.

(d) Prove that if
- $\varphi(v_1, \ldots, v_\ell)$ is a Π_2-formula in LST_A,
- $\xi < \lambda$ and $a_1, \ldots, a_\ell \in M_\xi$, and
- $(M_\xi, A_\xi) \models \varphi(\pi_{\xi,\xi'}(a_1), \ldots, \pi_{\xi,\xi'}(a_\ell))$ for all $\xi' > \xi$

then
$$\mathbb{D} \models \varphi(\pi'_\xi(a_1), \ldots, \pi'_\xi(a_\ell))$$

(e) Prove that E is extensional on D.

Now assume E is well-founded. Let (M, A) be the transitive collapse of $\mathbb{D}$ and let $k : \mathbb{D} \to (M, A)$ be the Mostowski collapsing isomorphism. For $\xi < \lambda$ let
$$\pi_\xi : (M_\xi, A_\xi) \to (M, A) \text{ be defined by } \pi_\xi = k \circ \pi'_\xi$$
It follows immediately that

(1) The statements analogous to (c) and (d) above with π_ξ in place of π'_ξ and (M, A) in place of $\mathbb{D}$ are true.

Also:

(2) $$D = \bigcup_{\xi<\lambda} \operatorname{rng}(\pi'_\xi) \quad \text{hence} \quad M = \bigcup_{\xi<\lambda} \operatorname{rng}(\pi).$$

Assume $\langle \tau_\xi \mid \xi < \lambda \rangle$ is a strictly increasing sequence of ordinals and $\tau_\xi = \mathsf{cr}(\pi_{\xi,\xi'})$ for $\xi < \xi'$. Notice that if this is true for one $\xi' > \xi$ then it is true for all $\xi' > \xi$. Let $\tau = \sup_{\xi<\lambda} \tau_\xi$.

(f) Prove that $\mathsf{cr}(\pi_\xi) = \tau_\xi$.

(g) Prove that if $\pi_{\xi,\xi'}(\tau_\xi) = \tau_{\xi'}$ whenever $\xi < \xi' < \lambda$ then $\pi_\xi(\tau_\xi) = \tau$ for all $\xi < \lambda$.

(i) Assume each M_ξ is rudimentarily closed/of the form $J^{B_\xi}_{\beta_\xi}$. Prove M is rudimentarily closed/of the form J^B_β.

Assume (N, D) is a transitive LST_A-structure and for every $\xi < \lambda$ we have a Σ_0-preserving map $\sigma_\xi : (M_\xi, A_\xi) \to (N, D)$ such that

$$\sigma_\xi = \sigma_{\xi'} \circ \pi_{\xi,\xi'} \quad \text{whenever } \xi < \xi'$$

That is, the diagram

$$\begin{tikzcd}
 & & & & (N,D) \\
(M_0,A_0) \arrow[r, "\pi_{0,1}"'] \arrow[urrrr, "\sigma_0"] & (M_1,A_1) \arrow[r, "\pi_{1,2}"'] \arrow[urrr, "\sigma_1"] & \cdots (M_\xi,A_\xi) \arrow[r, "\pi_{\xi,\xi+1}"'] \arrow[urr, "\sigma_\xi"] & \cdots \arrow[r] & (M,A)
\end{tikzcd}$$

commutes. Define a map $\sigma : (M, A) \to (N, D)$ by

$$\sigma(\pi_\xi(x)) = \sigma_\xi(x)$$

That is, σ completes the above diagram commutatively:

$$\begin{tikzcd}
 & & & & (N,D) \\
(M_0,A_0) \arrow[r, "\pi_{0,1}"'] \arrow[urrrr, "\sigma_0"] & (M_1,A_1) \arrow[r, "\pi_{1,2}"'] \arrow[urrr, "\sigma_1"] & \cdots (M_\xi,A_\xi) \arrow[r, "\pi_{\xi,\xi+1}"'] \arrow[urr, "\sigma_\xi"] & \cdots \arrow[r] & (M,A) \arrow[u, "\sigma"']
\end{tikzcd}$$

Prove the following.

(j) σ is well-defined, that is, its definition does not depend on the choice of ξ.

(k) σ is Σ_0-preserving.

(l) If all $\pi_{\xi,\xi'}$ are Σ_n-preserving then also all π_ξ are Σ_n-preserving, and if additionally all σ_ξ are Σ_n-preserving then also σ is Σ_n-preserving. Here $n \in \omega + 1$.

4. (1/2 page) Consider a commutative diagram as in Problem 3 with well-founded direct limit and assume that each (M_ξ, A_ξ) is an acceptable J-structure. Prove that (M, A) is an acceptable J-structure.

Assume further that for all $\xi < \xi' < \lambda$ the following hold.

(i) $\kappa \in M_\xi$ is an ordinal such that $\pi_{\xi,\xi'} \restriction \kappa = \mathsf{id} \restriction \kappa$,

(ii) $r_\xi \in M_\xi$ is such that $\pi_{\xi,\xi'}(r_\xi) = r_{\xi'}$, and

(iii) $Y_\xi = h_{(M_\xi,A_\xi)}(\kappa \cup \{r_\xi\})$ and $\iota_\xi : (\bar{M}_\xi, \bar{A}_\xi) \to (M_\xi, A_\xi)$ is the inverse of the Mostowski collapsing isomorphism coming from collapsing Y_ξ, and

(iv) $r = \pi_\xi(r_\xi)$ where notice r does not depend on ξ, $Y = h_{(M,A)}(\kappa \cup \{r\})$ and $\iota : (\bar{M}, \bar{A}) \to (M, A)$ is the inverse of the Mostowski collapsing isomorphism coming from collapsing Y.

Prove that if

$$\mathcal{P}(\kappa) \cap \bar{M}_\xi = \mathcal{P}(\kappa) \cap M_\xi \text{ for all } \xi < \lambda$$

then

$$\mathcal{P}(\kappa) \cap \bar{M} = \mathcal{P}(\kappa) \cap M$$

5. (1/2 page) Consider a commutative diagram as in Problem 3 with well-founded direct limit. Assume $\theta \in \mathbf{On}$ is such that:

(a) $\mathbf{On} \cap M_\xi = \theta$, and
(b) For every $\alpha < \theta$ there is a $\xi < \lambda$ such that $\mathsf{cr}(\pi_{\xi,\xi'}) > \pi_{0,\xi}(\alpha)$ whenever $\xi < \xi'$.

Prove that $\mathbf{On} \cap M = \theta$.

6. Consider a commutative diagram $\langle (M_\xi, A_\xi) \mid \xi < \lambda \rangle$ as in Problem 3 but this time allow M_ξ be a proper class for some (possibly all) $\xi < \lambda$. For the moment, still assume $\lambda \in \mathbf{On}$. We want the entire diagram be a class, and notice that writing "$\langle (M_\xi, A_\xi) \mid \xi < \lambda \rangle$" is not formally correct if M_ξ is a proper class. The correct way of expressing this is to require that

$$\{\langle \xi, x \rangle \mid x \in M_\xi\}$$

is a class. Similarly, to say that we have the sequence of maps $\langle \pi_{\xi,\xi'} \mid \xi \le \xi' < \lambda \rangle$ we require that

$$\{\langle \xi, \xi', x, y \rangle \mid \xi \le \xi' < \lambda \ \wedge \ \pi_{\xi,\xi'}(x) = y\}$$

is a class.

We can now define D_0 as before, and D_0 will be a class. We can also define I on D_0 as before, and again I will be class. Both will be proper classes iff at least one M_ξ is a proper class. Also as before, I will be an equivalence relation on D_0, but the equivalence classes now may be proper classes so the quotient D_0/I will not be a class if we define it in the usual way. One way of dealing with this situation is as follows (due to Kunen (?)). For $\langle \xi, x \rangle \in D_0$ let

$$\alpha(\langle \xi, x \rangle) = \min\{\mathsf{rank}(z) \mid z \in [\langle \xi, x \rangle]_I\}$$

and then set

$$[\xi, x]^* = [\langle \xi, x \rangle]_I \cap V_{\alpha(\langle \xi, x \rangle)}.$$

Notice that $[\xi, x]^*$ is well-defined, that is, it does not depend on the choice of representative. And, $[\xi, x]^*$ is a set. We now set

$$D = \{[\xi, x]^* \mid \langle \xi, x \rangle \in D_0\}$$

and consider D be the quotient D_0/I. On D we can then define a binary relation E the same way as before, and treat the unary predicate denoted by the symbol $\dot{A}$ in the obvious way. We can then prove the rest of the conclusions in Problem 3 more or less as before (Check for yourself!), but we need one additional step to make sure we can transitivize $\mathbb{D}$; this step is clause (A) below.

(A) **(1/3 page)** E is set-like.

(B) **(1/3 page)** Assume M_0 is a proper class. (So all M_ξ are proper classes.) Assume further that the direct limit of the diagram is well-founded. Prove that for every ordinal α there is an ordinal $\xi < \lambda$ such that $\pi_{\xi',\xi''}(\alpha) = \alpha$ whenever $\xi \leq \xi' \leq \xi'' < \lambda$.

Notice that (A) above was vacuously true in Problem 3, as we only dealt with diagrams which were sets. Clause (B) is a useful fact about commutative diagrams with well-founded direct limits. Notice that these are in particular all diagrams where $\mathsf{cf}(\lambda) > \omega$.

7. We now make the final generalization to the direct limits from Problem 3. So far we assumed the length λ of the diagram was an ordinal; now assume $\lambda = \mathbf{On}$. The new element here is that the diagram will be a proper class even if all M_ξ are sets. The approach from Problem 6 applies here and all clauses from Problem 3 can be dealt with this way, except one, namely:

E may fail to be set-like.

Thus, to be more precise, if E turns out to be set-like, we can proceed "as before". To understand that having E set-like is a serious issue, consider the following.

Consider a commutative diagram

$$\langle M_\xi, \pi_{\xi,\xi'} \mid \xi, \xi' \in \mathbf{On} \ \wedge \ \xi \leq \xi' \rangle$$

such that

(a) Each $M_\xi \in \mathbf{V}$ is a transitive model of ZFC^-.
(b) $\langle \kappa_\xi \mid \xi \in \mathbf{On} \rangle$ is a strictly increasing sequence of ordinals.
(c) Each $\pi_{\xi,\xi'} : M_\xi \to M_{\xi'}$ is fully elementary.
(d) $\mathsf{cr}(\pi_{\xi,\xi'}) = \kappa_\xi$ and $\pi_{\xi,\xi'}(\kappa_\xi) = \kappa_{\xi'}$ whenever $\xi < \xi'$.

Prove **(1/3 page)** that for this diagram, the relation E is not set-like, but it is well-founded. So this is an example of well-founded relation which is extensional but cannot be transitivized.

Remark. We will see later that a diagram as above can be constructed, and even with $M_\xi = J_{\tau_\xi}$ for suitable τ_ξ.