MATH 281C SPRING 2020 HOMEWORK 2

Target date: Tuesday, May 19, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset.

(a) Assume now $\mathbb{P}$ has a largest element $1_{\mathbb{P}}$. Recall that for $a \in \mathbf{V}$ we defined
$$\check{a} = \{\langle 1_{\mathbb{P}}, \check{z}\rangle \mid z \in a\}$$
and
$$\dot{G} = \{\langle p, \check{p}\rangle \mid p \in \mathbb{P}\}$$
Prove that for every $(\mathbb{P}, \mathbf{V})$-generic filter H we have $\check{a}^H = a$ and $\dot{G}^H = H$.

(b) Now assume $\mathbb{P}$ does not have a largest element. Try to define $\mathbb{P}$-terms $\check{a}$ for $a \in \mathbf{V}$ and also $\dot{G}$ so that we still have $\check{a}^H = a$ and $\dot{G}^H = H$ for every $(\mathbb{P}, \mathbf{V})$-generic filter H.

2. (1/2 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset and $x \in \mathbf{V}$.

(a) Construct a $\mathbb{P}$-term $\dot{x}$ such that $x^G = \dot{x}^G$ for all $(\mathbb{P}, \mathbf{V})$-generic filters G.

(b) Prove that $\mathsf{rank}(x^G) \leq \mathsf{rank}(x)$ whenever G is a $(\mathbb{P}, \mathbf{V})$-generic filter.

3. (1/2 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset and $\dot{x}, \dot{y} \in \mathbf{V}$ are $\mathbb{P}$-terms.

(a) Write down a $\mathbb{P}$-term $\dot{a}$ such that for every $(\mathbb{P}, \mathbf{V})$-generic filter G we have $\dot{a}^G = \{\dot{x}^G, \dot{y}^G\}$.

(b) Write down a $\mathbb{P}$-term $\dot{b}$ such that for every $(\mathbb{P}, \mathbf{V})$-generic filter G we have $\dot{b} = \bigcup \dot{x}^G$.

(c) Assume $p, q \in \mathbb{P}$ are incompatible. Prove that for every $(\mathbb{P}, \mathbf{V})$-generic filter G with $p \in G$ there is a proper class of $\mathbb{P}$-terms $\dot{x}$ such that $\dot{x}^G = \varnothing$.

Exercises 4 – 6 are close to some exercises we did in connection with forcing axioms.

4. (1/2 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset. Prove that (a),(b) and (c) are equivalent and each of them implies (d). Working in ZFC, prove that (d) implies (a),(b) and (c).

(a) G is $(\mathbb{P}, \mathbf{V})$-generic.

(b) $G \cap D \neq \varnothing$ whenever $D \in \mathbf{V}$ is a predense subset of $\mathbb{P}$.

(c) $G \cap D \neq \varnothing$ whenever $D \in \mathbf{V}$ is an open dense subset of $\mathbb{P}$.

(d) $G \cap D \neq \varnothing$ whenever $D \in \mathbf{V}$ is a maximal antichain in $\mathbb{P}$.

5. (1/2 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset, $B \in \mathbf{V}$ is an open subset of $\mathbb{P}$ and G is a $(\mathbb{P}, \mathbf{V})$-generic filter such that $G \cap B \neq \varnothing$.

(a) Assume $D \in \mathbf{V}$ and $D \subseteq B$ is dense in B, that is, for every $p \in B$ there is some $q \in D$ such that $q \leq p$. Prove that $G \cap A \neq \varnothing$.

(b) Assume $A \in \mathbf{V}$ and $A \subseteq B$ is predense in B, that is, for every $p \in B$ there is some $q \in A$ such that p, q are compatible. Prove that $G \cap A \neq \varnothing$.

In particular, the conclusion applies to A which is an antichain in $\mathbb{P}$ and is maximal among all antichains in $\mathbf{V}$ contained in B, that is, if $A' \in \mathbf{V}$ is such that $A' \subseteq B$ is an antichain in $\mathbb{P}$ and $A \subseteq A'$ then $A = A'$.

6. (1/3 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset, and $G \subseteq \mathbb{P}$ satisfies the following:

(a) G is upward closed,

(b) Any two elements of G are compatible in $\mathbb{P}$, but we do not require there is a lower bound for these two elements in G, and

(c) $G \cap D \neq \varnothing$ whenever $D \in \mathbf{V}$ is a dense subset of $\mathbb{P}$.

Prove that G is a $(\mathbb{P}, \mathbf{V})$-generic filter.

7. (2/3 page) Work in ZF. Assume $\mathbb{P} \in \mathbf{V}$ is a poset. We say that $\mathbb{P}$ is **separative** iff the following holds for every $p, q \in \mathbb{P}$.

($*$) If $q \not\leq p$ then there is a condition $r \leq q$ such that $r \perp p$.

Consider the following posets.

(a) $\mathbb{P} = (\mathsf{FN}(A, B, \kappa), \supseteq)$ where, assuming κ is an infinite cardinal and A, B are sets such that A is well-orderable with $\mathsf{card}(A) \geq \kappa$,

$$\mathsf{FN}(A, B, \kappa) = \{f : a \to B \mid a \in [A]^{<\kappa}\}$$

(b) Assume $\mathcal{I}$ is an ideal on an infinite set Z extending

$$\mathcal{I}_\omega(Z) = \{X \subseteq Z \mid X \text{ is finite}\}.$$

Let $\mathbb{P} = (\mathcal{I}^+, \subseteq)$.

(c) Given a Boolean algebra $\mathbb{B}$, let $\mathbb{P} = (\mathbb{B} \setminus \{0\}, \leq_{\mathbb{B}})$.

(d) Given is a sequence of separative posets $\langle \mathbb{P}_n \mid n \in \omega \rangle$, and assume each $\mathbb{P}_n$ has a largest element $1_{\mathbb{P}_n}$. Define a poset $\mathbb{P}$ as follows.

- Conditions in $\mathbb{P}$ are finite functions p with $\mathrm{dom}(p) \subseteq \omega$ such that $p(n) \in \mathbb{P}_n$ for each $n \in \mathrm{dom}(p)$.
- Ordering on $\mathbb{P}$ is defined as follows: $p \leq q$ iff $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(n) \leq_{\mathbb{P}_n} q(n)$ for every $n \in \mathrm{dom}(q)$.

For each of the posets in (a) – (d) determine whether it is separative.