MATH 281C SPRING 2020 HOMEWORK 3

Target date: Tuesday, May 26, 2020

Rules: Write as efficiently as possible. Include all relevant points, but do not write too much. Think carefully what to write and what not in order to make the presentation of your argument clear and reasonable. Use common sense to determine the amount of details that need to be included, and keep in mind that your text should correspond to graduate level.

Quote any result from the lecture that comes up in your argument: Do not re-prove these results. If the statement of a problem indicates the maximum allowed length, this length is much more than needed. If you type, do not use font smaller than 10pt.

I will not grade any text that exceeds the specified length.

1. (1/2 page) Work in ZFC. Let κ be a regular cardinal and $\mathbb{P}$ be a poset. Prove the following.

(a) If $\mathbb{P}$ is κ-closed then $\mathbb{P}$ is strategically κ-closed.

(b) If $\mathbb{P}$ is strategically κ-closed then $\mathbb{P}$ is (κ, ∞)-distributive.

We will write briefly "κ-distributive" for "(κ, ∞)-distributive".

2. (2/3 page) Work in ZFC. Let $\mathbb{P}$ be a poset.

(a) Assume $A \in \mathbf{V}$, $\dot{x}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ is a condition such that

$$p \Vdash (\dot{x} \subseteq \check{A} \ \wedge \ \dot{x} \notin \mathbf{V}).$$

Prove that for every $q \leq p$ there is some $a \in A$ such that

$$q \not\Vdash \check{a} \in \dot{x} \ \wedge \ q \not\Vdash \check{a} \notin \dot{x}$$

(b) Let κ be regular and $\gamma < \kappa$. Assume $\mathbb{P}$ is κ-distributive, $\dot{x}$ is a $\mathbb{P}$-term and $p \in \mathbb{P}$ is a condition such that

$$p \Vdash \dot{x} \subseteq \check{\gamma}.$$

Prove that there is a set $a \in \mathbf{V}$ such that $a \subseteq \gamma$ and a condition $q \leq p$ such that

$$q \Vdash \dot{x} = \check{a}.$$

3. (1/2 page) Work in ZFC. Let κ be an uncountable regular cardinal and S_G be the canonical Cohen subset of κ in the generic extension $\mathbf{V}[G]$. More precisely, let

- $\mathbb{P} = \mathsf{Add}(\kappa, 1)$,
- G be a $(\mathbb{P}, \mathbf{V})$-generic filter,
- $g = \bigcup G : \kappa \to \{0, 1\}$, and
- $S_G = \{\xi < \kappa \mid g(\xi) = 1\}$.

Prove that $\mathbf{V}[G] \models S$ is stationary.

4. (1/2 page) Work in ZFC. Let κ be a regular cardinal, $\mathbb{P}$ be a κ-c.c. poset, and G be a $(\mathbb{P}, \mathbf{V})$-generic filter. Prove the following.

(a) $C \in \mathbf{V}[G]$ is a club in κ then there is some club C^* in κ such that $C^* \in \mathbf{V}$ and $C^* \subseteq C$.

(b) If $S \subseteq \kappa$ is stationary in κ in the sense of $\mathbf{V}$ then S is stationary in the sense of $\mathbf{V}[G]$.

5. (2/3 page) Work in ZFC. Let κ be a regular cardinal, $\mathbb{P}$ be a strategically κ-closed poset, and G be a $(\mathbb{P}, \mathbf{V})$-generic filter.

(a) Consider the following poset $\mathbb{P}$.
- **Conditions** in $\mathbb{P}$ are closed bounded subsets of ω_1. For each $c \in \mathbb{P}$ let $\gamma_c = \max(c)$.
- **Ordering** in $\mathbb{P}$ is end-extension: $c \leq c' \iff c' = c \cap (\gamma_{c'} + 1)$

Prove the following:

(i) $\mathbb{P}$ is ω_1-closed, so in particular it preserves ω_1.

(ii) Let G be a $(\mathbb{P}, \mathbf{V})$-generic filter and $C_G = \bigcup G$. Prove that C_G is a club subset of ω_1 such that $C_G \triangle C$ is an unbounded subset of ω_1 whenever $C \in \mathbf{V}$ is a club in ω_1. Compare with Problem 1(b).

(b) Prove that if $S \subseteq \kappa$ is stationary in κ in the sense of $\mathbf{V}$ then S is stationary in the sense of $\mathbf{V}[G]$.

6. (1/2 page) Work in ZFC. Let κ be strongly inaccessible and T be the complete binary tree of height κ. That is,

$$T = ({}^{<\kappa}\{0, 1\}, \subset).$$

Let $\mathbb{P} = \mathsf{Coll}(\omega, < \kappa)$ and G be a $(\mathbb{P}, \mathbf{V})$-generic filter. Prove the following.

(a) $\mathbb{P}$ has the κ-Knaster property. (Actually for (a) we only need that κ is regular, so come up with a proof that works for any regular κ.)

(b) $\omega_1^{\mathbf{V}[G]} = \kappa$.

(c) $\mathbf{V}[G] \models T$ is a Kurepa tree.

7. (1/2 page) Work in ZF. Let $\mathbb{P}$ be a poset. Define a binary relation $\leq^*$ on $\mathbb{P}$ by

$$p \leq^* q \iff (\forall r \leq p)(r \,||\, q)$$

Prove the following.

(a) $\leq^*$ is reflexive and transitive, so $\mathbb{P}^* = (\mathbb{P}, \leq^*)$ is a poset.

(b) $\leq \subseteq \leq^*$. If $\mathbb{P}$ is separative then $\leq^* = \leq$, so $\mathbb{P}^* = \mathbb{P}$.

(c) For $p, q \in \mathbb{P}$ the following are equivalent:

(i) $p \leq^* q$

(ii) $p \Vdash \check{q} \in \dot{G}$

where recall that $\dot{G}$ is the canonical $\mathbb{P}$-name for the $(\mathbb{P}, \mathbf{V})$-generic filter.

(d) $\mathbb{P}^*$ is separative. (See HW 2, Problem 7.)

(e) The identity map $\mathsf{id} : \mathbb{P} \to \mathbb{P}^*$ is a regular embedding.

The poset $\mathbb{P}^*$ is called the **separative quotient** of $\mathbb{P}$. Since the identity map $\mathsf{id} : \mathbb{P} \to \mathbb{P}^*$ is trivially surjective, by (d) above, it is a dense embedding. It follows that $\mathbb{P}$ and $\mathbb{P}^*$ give the same generic extensions, so any generic extension can be obtained via a separative poset.